EXPRESSWALA ACADEMY

Maths 1A: Inverse Trigonometric Functions

Question Paper, Set 03

Time: 1 hour 30 minutes | Maximum marks: 50

Name: _____ Class: _____ Date: _____

1. Answer all 20 questions. The marks for each question are shown in brackets.
2. Show your working for the 2, 3 and 5 mark questions.
3. The key sheet and worked solutions are at the end of this paper.

1. Find the domain of $\sin^{-1}(4x+1)$. [1 mark]

2. Find the principal value of $\cos^{-1}\left(-\frac{1}{\sqrt{2}}\right)$. [1 mark]

3. Evaluate $\cos^{-1}\frac{\sqrt{3}}{2}+\sin^{-1}\frac{1}{\sqrt{2}}$. [2 marks]

4. Find $\tan^{-1}\left(\tan-\frac{\pi}{5}\right)$. [1 mark]

5. Using the rules for negative arguments, evaluate $\cos^{-1}(-1)+\sin^{-1}\left(-\frac{1}{\sqrt{2}}\right)$. [2 marks]

6. Express $\sec^{-1}\frac{9}{2}$ as an inverse cosine. [1 mark]

7. If $\tan^{-1}x=\frac{3\pi}{11}$, find $\cot^{-1}x$. [2 marks]

8. Prove that $\sin^{-1}\frac{12}{13}+\sin^{-1}\frac{3}{5}=\pi-\sin^{-1}\frac{63}{65}$. [5 marks]

9. Find $\tan^{-1}\frac{2}{3}+\tan^{-1}\frac{1}{5}$. [3 marks]

10. Solve $\tan^{-1}4x+\tan^{-1}6x=\frac{\pi}{4}$. [5 marks]

11. Prove that $\tan^{-1}\frac{2}{9}+\tan^{-1}\frac{1}{4}+\tan^{-1}\frac{1}{3}=\frac{\pi}{4}$. [5 marks]

12. Express $\tan^{-1}\frac{5}{12}$ as an inverse sine and as an inverse cosine. [2 marks]

13. Express $2\tan^{-1}\frac{2}{5}$ as a single inverse tangent, a single inverse sine and a single inverse cosine. [3 marks]

14. Write the end points of the graph of $y=\sin^{-1}x$. [1 mark]

15. Find $\tan^{-1}(\tan 2)$. [2 marks]

16. Find the value of $\cos^{-1}(4x^3-3x)$ when $x=\frac{1}{2}$. [3 marks]

17. If $3\sin^{-1}x-2\cos^{-1}x=\pi$, find $3\sin^{-1}x+2\cos^{-1}x$. [3 marks]

18. If $\sin^{-1}\frac{x}{3}+\sin^{-1}\frac{y}{4}=\theta$, prove that $16x^2+24xy\cos\theta+9y^2=144\sin^2\theta$. [5 marks]

19. Which substitution removes the square root in $\sqrt{25+x^2}$? What does the root become? [1 mark]

20. Simplify $\tan^{-1}\frac{\sqrt{x^2-25}}{5}$ for $x \geq 5$. [2 marks]

Key sheet

Q	Answer
1	$[-\frac{1}{2}, 0]$
2	$\frac{3\pi}{4}$
3	$\frac{5\pi}{12}$
4	$-\frac{\pi}{5}$
5	$\frac{3\pi}{4}$
6	$\cos^{-1}\frac{2}{9}$
7	$\frac{5\pi}{22}$
8	Proved: $\sin^{-1}\frac{12}{13} + \sin^{-1}\frac{3}{5} = \pi - \sin^{-1}\frac{63}{65}$
9	$\frac{\pi}{4}$
10	$x = \frac{1}{12}$
11	Proved
12	$\sin^{-1}\frac{5}{13}$, $\cos^{-1}\frac{12}{13}$
13	$\tan^{-1}\frac{20}{21}$, $\sin^{-1}\frac{20}{29}$, $\cos^{-1}\frac{21}{29}$
14	$\left(-1, -\frac{\pi}{2}\right)$ and $\left(1, \frac{\pi}{2}\right)$
15	$2 - \pi$
16	π
17	$\frac{7\pi}{5}$
18	Proved
19	$x = 5\tan\theta$; the root becomes $5\sec\theta$
20	$\sec^{-1}\frac{x}{5}$

Solutions

1. Find the domain of $\sin^{-1}(4x+1)$. [1]

$-1 \le 4x+1 \le 1$

$-2 \le 4x \le 0$

$-\frac{1}{2} \le x \le 0$

Ans: $[-\frac{1}{2}, 0]$

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2. Find the principal value of $\cos^{-1}\left(-\frac{1}{\sqrt{2}}\right)$. [1]

We need $\theta \in [0, \pi]$ with $\cos\theta = -\frac{1}{\sqrt{2}}$.

$\therefore \cos^{-1}\left(-\frac{1}{\sqrt{2}}\right) = \frac{3\pi}{4}$

Ans: $\frac{3\pi}{4}$

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3. Evaluate $\cos^{-1}\frac{\sqrt{3}}{2} + \sin^{-1}\frac{1}{\sqrt{2}}$. [2]

$\cos^{-1}\frac{\sqrt{3}}{2} = \frac{\pi}{6},\ \sin^{-1}\frac{1}{\sqrt{2}} = \frac{\pi}{4}$

$\frac{\pi}{6} + \frac{\pi}{4} = \frac{5\pi}{12}$

Ans: $\frac{5\pi}{12}$

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4. Find $\tan^{-1}\left(\tan -\frac{\pi}{5}\right)$. [1]

$-\frac{\pi}{5}$ lies in the principal range $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

So $\tan^{-1}(\tan\theta) = \theta$ gives $-\frac{\pi}{5}$.

Ans: $-\frac{\pi}{5}$

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5. Using the rules for negative arguments, evaluate $\cos^{-1}(-1) + \sin^{-1}\left(-\frac{1}{\sqrt{2}}\right)$. [2]

$\cos^{-1}(-1) = \pi - \cos^{-1}1 = \pi - 0 = \pi$

$\sin^{-1}\left(-\frac{1}{\sqrt{2}}\right) = -\sin^{-1}\frac{1}{\sqrt{2}} = -\frac{\pi}{4}$

$\text{Sum} = \frac{3\pi}{4}$

Ans: $\frac{3\pi}{4}$

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6. Express $\sec^{-1}\frac{9}{2}$ as an inverse cosine. [1]

$\sec^{-1}x = \cos^{-1}\frac{1}{x},\ |x| \ge 1$

$\sec^{-1}\frac{9}{2} = \cos^{-1}\frac{2}{9}$

Ans: $\cos^{-1}\frac{2}{9}$

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7. If $\tan^{-1}x = \frac{3\pi}{11}$, find $\cot^{-1}x$. [2]

$\tan^{-1}x + \cot^{-1}x = \frac{\pi}{2}$

$\cot^{-1}x = \frac{\pi}{2} - \frac{3\pi}{11} = \frac{5\pi}{22}$

Ans: $\frac{5\pi}{22}$

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8. Prove that $\sin^{-1}\frac{12}{13}+\sin^{-1}\frac{3}{5}=\pi-\sin^{-1}\frac{63}{65}$. [5]

Let $x=\frac{12}{13}$, $y=\frac{3}{5}$.

$\sqrt{1-x^2}=\frac{5}{13},\ \sqrt{1-y^2}=\frac{4}{5}$

$x^2+y^2=\frac{5121}{4225}>1$ and both are positive.

$\pi-\sin^{-1}\left(x\sqrt{1-y^2}+y\sqrt{1-x^2}\right)=\pi-\sin^{-1}\left(\frac{12}{13}\cdot\frac{4}{5}+\frac{3}{5}\cdot\frac{5}{13}\right)=\pi-\sin^{-1}\frac{63}{65}$

Hence $\sin^{-1}\frac{12}{13}+\sin^{-1}\frac{3}{5}=\pi-\sin^{-1}\frac{63}{65}$.

Ans: Proved: $\sin^{-1}\frac{12}{13}+\sin^{-1}\frac{3}{5}=\pi-\sin^{-1}\frac{63}{65}$

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9. Find $\tan^{-1}\frac{2}{3}+\tan^{-1}\frac{1}{5}$. [3]

Here $xy=\frac{2}{15}<1$.

$\frac{x+y}{1-xy}=\frac{\frac{13}{15}}{\frac{13}{15}}=1$

$\tan^{-1}\frac{2}{3}+\tan^{-1}\frac{1}{5}=\tan^{-1}1=\frac{\pi}{4}$

Ans: $\frac{\pi}{4}$

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10. Solve $\tan^{-1}4x+\tan^{-1}6x=\frac{\pi}{4}$. [5]

$\tan^{-1}\frac{10x}{1-24x^2}=\frac{\pi}{4}\Rightarrow\frac{10x}{1-24x^2}=1$

$24x^2+10x-1=0$

$x=-\frac{1}{2}$ or $x=\frac{1}{12}$

For $x=-\frac{1}{2}$ both inverse tangents are negative, so the left side is negative and cannot equal $\frac{\pi}{4}$; reject it.

$\therefore x=\frac{1}{12}$

Ans: $x=\frac{1}{12}$

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11. Prove that $\tan^{-1}\frac{2}{9}+\tan^{-1}\frac{1}{4}+\tan^{-1}\frac{1}{3}=\frac{\pi}{4}$. [5]

Let $x=\frac{2}{9}$, $y=\frac{1}{4}$, $z=\frac{1}{3}$, all positive.

$S_1=x+y+z=\frac{29}{36},\ S_2=xy+yz+zx=\frac{23}{108}<1,\ S_3=xyz=\frac{1}{54}$

$\tan^{-1}x+\tan^{-1}y+\tan^{-1}z=\tan^{-1}\frac{S_1-S_3}{1-S_2}=\tan^{-1}\frac{\frac{85}{108}}{\frac{85}{108}}$

$=\tan^{-1}1=\frac{\pi}{4}$

Ans: Proved

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12. Express $\tan^{-1}\frac{5}{12}$ as an inverse sine and as an inverse cosine. [2]

$\sqrt{1+x^2}=\frac{13}{12}$

$\tan^{-1}\frac{5}{12}=\sin^{-1}\frac{x}{\sqrt{1+x^2}}=\sin^{-1}\frac{5}{13}$

$\tan^{-1}\frac{5}{12}=\cos^{-1}\frac{1}{\sqrt{1+x^2}}=\cos^{-1}\frac{12}{13}$

Ans: $\sin^{-1}\frac{5}{13}$, $\cos^{-1}\frac{12}{13}$

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13. Express $2\tan^{-1}\frac{2}{5}$ as a single inverse tangent, a single inverse sine and a single inverse cosine. [3]

$0 < x = \frac{2}{5} < 1$.

$2\tan^{-1}x = \tan^{-1}\frac{2x}{1-x^2} = \tan^{-1}\frac{\frac{4}{5}}{\frac{21}{25}} = \tan^{-1}\frac{20}{21}$

$2\tan^{-1}x = \sin^{-1}\frac{2x}{1+x^2} = \sin^{-1}\frac{\frac{4}{5}}{\frac{29}{25}} = \sin^{-1}\frac{20}{29}$

$2\tan^{-1}x = \cos^{-1}\frac{1-x^2}{1+x^2} = \cos^{-1}\frac{\frac{21}{25}}{\frac{29}{25}} = \cos^{-1}\frac{21}{29}$

Ans: $\tan^{-1}\frac{20}{21}$, $\sin^{-1}\frac{20}{29}$, $\cos^{-1}\frac{21}{29}$

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14. Write the end points of the graph of $y = \sin^{-1}x$. [1]

The domain is $[-1, 1]$ and $\sin^{-1}(\pm 1) = \pm\frac{\pi}{2}$.

Ans: $\left(-1, -\frac{\pi}{2}\right)$ **and** $\left(1, \frac{\pi}{2}\right)$

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15. Find $\tan^{-1}(\tan 2)$. [2]

2 is not in the principal range $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

Using the periodic graph of $\tan^{-1}(\tan x)$, shift or reflect it into the range: the value is $2 - \pi$.

Check: $2 - \pi$ is in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ and $\tan(2 - \pi) = \tan(2)$.

Ans: $2 - \pi$

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16. Find the value of $\cos^{-1}(4x^3 - 3x)$ when $x = \frac{1}{2}$. [3]

$x = \frac{1}{2}$ lies in $-\frac{1}{2} \le x \le \frac{1}{2}$, where $\cos^{-1}(4x^3 - 3x) = 2\pi - 3\cos^{-1}x$.

$= \pi$

Ans: π

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17. If $3\sin^{-1}x - 2\cos^{-1}x = \pi$, find $3\sin^{-1}x + 2\cos^{-1}x$. [3]

Put $\cos^{-1}x = \frac{\pi}{2} - \sin^{-1}x$:

$3\sin^{-1}x - 2\left(\frac{\pi}{2} - \sin^{-1}x\right) = \pi \Rightarrow \sin^{-1}x = \frac{2\pi}{5}$

$3\sin^{-1}x + 2\cos^{-1}x = 3\cdot\frac{2\pi}{5} + 2\left(\frac{\pi}{2} - \frac{2\pi}{5}\right) = \frac{7\pi}{5}$

Ans: $\frac{7\pi}{5}$

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18. If $\sin^{-1}\frac{x}{3} + \sin^{-1}\frac{y}{4} = \theta$, prove that $16x^2 + 24xy\cos\theta + 9y^2 = 144\sin^2\theta$. [5]

Let $\sin^{-1}\frac{x}{3} = \alpha$, $\sin^{-1}\frac{y}{4} = \beta$, so $\alpha + \beta = \theta$.

Then $\sin\alpha = \frac{x}{3}$, $\cos\alpha = \sqrt{1 - \frac{x^2}{9}}$, and similarly for β.

$\cos\theta = \cos\alpha\cos\beta - \sin\alpha\sin\beta = \sqrt{1-\frac{x^2}{9}}\sqrt{1-\frac{y^2}{16}} - \frac{xy}{12}$

$\left(\cos\theta + \frac{xy}{12}\right)^2 = \left(1 - \frac{x^2}{9}\right)\left(1 - \frac{y^2}{16}\right)$

$\cos^2\theta + \frac{2xy}{12}\cos\theta + \frac{x^2y^2}{144} = 1 - \frac{x^2}{9} - \frac{y^2}{16} + \frac{x^2y^2}{144}$

$\frac{x^2}{9} + \frac{2xy}{12}\cos\theta + \frac{y^2}{16} = \sin^2\theta$

Multiplying by 144: $16x^2 + 24xy\cos\theta + 9y^2 = 144\sin^2\theta$.

Ans: Proved

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19. Which substitution removes the square root in $\sqrt{25 + x^2}$? What does the root become? [1]

For $\sqrt{25 + x^2}$ put $x = 5\tan\theta$.

Then the root becomes $5\sec\theta$.

Ans: $x = 5\tan\theta$**; the root becomes** $5\sec\theta$

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20. Simplify $\tan^{-1}\frac{\sqrt{x^2-25}}{5}$ for $x \geq 5$. [2]

Put $x = 5\sec\theta$, $\theta \in \left[0, \frac{\pi}{2}\right)$; then $\sqrt{x^2-25} = 5\tan\theta$.

$\tan^{-1}(\tan\theta) = \theta = \sec^{-1}\frac{x}{5}$

Ans: $\sec^{-1}\frac{x}{5}$