EXPRESSWALA ACADEMY

Maths 1A: Inverse Trigonometric Functions

Question Paper, Set 02

Time: 1 hour 30 minutes | Maximum marks: 50

Name: _____ Class: _____ Date: _____

1. Answer all 20 questions. The marks for each question are shown in brackets.
2. Show your working for the 2, 3 and 5 mark questions.
3. The key sheet and worked solutions are at the end of this paper.

1. For what values of x is $\cos^{-1}(2x+3)$ defined? [1 mark]

2. Find the principal value of $\tan^{-1}\left(-\sqrt{3}\right)$. [1 mark]

3. Evaluate $\tan^{-1} 1 - \cot^{-1}\sqrt{3}$. [2 marks]

4. Find $\cos\left(\cos^{-1}\frac{2}{7}\right)$. [1 mark]

5. Using the rules for negative arguments, evaluate $\tan^{-1}\left(-\frac{1}{\sqrt{3}}\right) + \sec^{-1}\left(-\frac{2}{\sqrt{3}}\right)$. [2 marks]

6. Express $\cot^{-1}\left(-\frac{5}{2}\right)$ in terms of an inverse tangent. [1 mark]

7. Solve $\sin^{-1} x = 2\cos^{-1} x$. [2 marks]

8. Prove that $\cos^{-1}\frac{3}{5} + \cos^{-1}\frac{8}{17} = \cos^{-1}\left(-\frac{36}{85}\right)$. [5 marks]

9. Find $\tan^{-1} 4 + \tan^{-1} 5$. [3 marks]

10. Solve $\tan^{-1}(x+1) + \tan^{-1}(x-1) = \tan^{-1} 2$. [5 marks]

11. Prove that $\tan^{-1}\frac{1}{2} + \tan^{-1}\frac{1}{4} + \tan^{-1}\frac{1}{13} = \frac{\pi}{4}$. [5 marks]

12. Express $\cos^{-1}\left(-\frac{5}{13}\right)$ as an inverse sine. [2 marks]

13. Express $2\tan^{-1} 3$ as a single inverse tangent, a single inverse sine and a single inverse cosine. [3 marks]

14. Write the horizontal asymptotes of the graph of $y = \tan^{-1} x$. [1 mark]

15. Find $\cos^{-1}(\cos 5)$. [2 marks]

16. Simplify $\tan^{-1}\frac{2x}{1-x^2}$ for $x > 1$. [3 marks]

17. If $\sin^{-1}\frac{8}{x} + \sin^{-1}\frac{15}{x} = \frac{\pi}{2}$ with $x > 0$, find x. [3 marks]

18. Prove that $\tan^{-1}\frac{1}{1+2\cdot 3} + \tan^{-1}\frac{1}{1+3\cdot 4} + \cdots + \tan^{-1}\frac{1}{1+(n+1)(n+2)} = \tan^{-1}\frac{n}{1+2(n+2)}$. [5 marks]

19. Which substitution removes the square root in $\sqrt{64+x^2}$? What does the root become? [1 mark]

20. Simplify $\sin^{-1}\frac{x}{\sqrt{4+x^2}}$ for $x \in \mathbb{R}$. [2 marks]

Key sheet

Q	Answer
1	$[-2, -1]$
2	$-\frac{\pi}{3}$
3	$\frac{\pi}{12}$
4	$\frac{2}{7}$
5	$\frac{2\pi}{3}$
6	$\pi - \tan^{-1}\frac{2}{5}$
7	$x = \frac{\sqrt{3}}{2}$
8	Proved: $\cos^{-1}\frac{3}{5} + \cos^{-1}\frac{8}{17} = \cos^{-1}\left(-\frac{36}{85}\right)$
9	$\pi + \tan^{-1}\left(-\frac{9}{19}\right)$
10	$x = 1$
11	Proved
12	$\pi - \sin^{-1}\frac{12}{13}$
13	$\pi - \tan^{-1}\frac{3}{4}$, $\pi - \sin^{-1}\frac{3}{5}$, $\cos^{-1}\left(-\frac{4}{5}\right)$
14	$y = \frac{\pi}{2}$ and $y = -\frac{\pi}{2}$
15	$-5 + 2\pi$
16	$-\pi + 2\tan^{-1}x$
17	$x = 17$
18	Proved
19	$x = 8\tan\theta$; the root becomes $8\sec\theta$
20	$\tan^{-1}\frac{x}{2}$

Solutions

1. For what values of x is $\cos^{-1}(2x+3)$ defined? [1]

$-1 \le 2x+3 \le 1$

$-4 \le 2x \le -2$

$-2 \le x \le -1$

Ans: $[-2, -1]$

---

2. Find the principal value of $\tan^{-1}\left(-\sqrt{3}\right)$. [1]

We need $\theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ with $\tan\theta = -\sqrt{3}$.

$\therefore \tan^{-1}\left(-\sqrt{3}\right) = -\frac{\pi}{3}$

Ans: $-\frac{\pi}{3}$

---

3. Evaluate $\tan^{-1} 1 - \cot^{-1}\sqrt{3}$. [2]

$\tan^{-1} 1 = \frac{\pi}{4},\ \cot^{-1}\sqrt{3} = \frac{\pi}{6}$

$\frac{\pi}{4} - \frac{\pi}{6} = \frac{\pi}{12}$

Ans: $\frac{\pi}{12}$

---

4. Find $\cos\left(\cos^{-1}\frac{2}{7}\right)$. [1]

$\cos(\cos^{-1}x) = x$ for every x in the domain of $\cos^{-1}$.

$\frac{2}{7}$ is in the domain, so the value is $\frac{2}{7}$.

Ans: $\frac{2}{7}$

---

5. Using the rules for negative arguments, evaluate $\tan^{-1}\left(-\frac{1}{\sqrt{3}}\right) + \sec^{-1}\left(-\frac{2}{\sqrt{3}}\right)$. [2]

$\tan^{-1}\left(-\frac{1}{\sqrt{3}}\right) = -\tan^{-1}\frac{1}{\sqrt{3}} = -\frac{\pi}{6}$

$\sec^{-1}\left(-\frac{2}{\sqrt{3}}\right) = \pi - \sec^{-1}\frac{2}{\sqrt{3}} = \pi - \frac{\pi}{6} = \frac{5\pi}{6}$

$\text{Sum} = \frac{2\pi}{3}$

Ans: $\frac{2\pi}{3}$

---

6. Express $\cot^{-1}\left(-\frac{5}{2}\right)$ in terms of an inverse tangent. [1]

$\cot^{-1}x = \pi + \tan^{-1}\frac{1}{x},\ x<0$

$\cot^{-1}\left(-\frac{5}{2}\right) = \pi + \tan^{-1}\left(-\frac{2}{5}\right) = \pi - \tan^{-1}\frac{2}{5}$

Ans: $\pi - \tan^{-1}\frac{2}{5}$

---

7. Solve $\sin^{-1}x = 2\cos^{-1}x$. [2]

Let $\cos^{-1}x = t$; then $\sin^{-1}x = \frac{\pi}{2} - t$.

$\frac{\pi}{2} - t = 2t \Rightarrow t = \frac{\pi}{6}$

$\cos^{-1}x = \frac{\pi}{6} \Rightarrow x = \cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}$

Ans: $x = \frac{\sqrt{3}}{2}$

---

8. Prove that $\cos^{-1}\frac{3}{5}+\cos^{-1}\frac{8}{17}=\cos^{-1}\left(-\frac{36}{85}\right)$. [5]

Let $x=\frac{3}{5}$, $y=\frac{8}{17}$.

$\sqrt{1-x^2}=\frac{4}{5},\ \sqrt{1-y^2}=\frac{15}{17}$

Here $x+y\geq 0$.

$xy-\sqrt{1-x^2}\sqrt{1-y^2}=\frac{3}{5}\cdot\frac{8}{17}-\frac{4}{5}\cdot\frac{15}{17}=-\frac{36}{85}$

So the formula gives $\cos^{-1}\left(-\frac{36}{85}\right)$.

Hence $\cos^{-1}\frac{3}{5}+\cos^{-1}\frac{8}{17}=\cos^{-1}\left(-\frac{36}{85}\right)$.

Ans: Proved: $\cos^{-1}\frac{3}{5}+\cos^{-1}\frac{8}{17}=\cos^{-1}\left(-\frac{36}{85}\right)$

---

9. Find $\tan^{-1}4+\tan^{-1}5$. [3]

Here $x,y>0,\ xy=20>1$.

$\frac{x+y}{1-xy}=\frac{9}{-19}=-\frac{9}{19}$

$\tan^{-1}4+\tan^{-1}5=\pi+\tan^{-1}\left(-\frac{9}{19}\right)$

Ans: $\pi+\tan^{-1}\left(-\frac{9}{19}\right)$

---

10. Solve $\tan^{-1}(x+1)+\tan^{-1}(x-1)=\tan^{-1}2$. [5]

$\tan^{-1}\frac{2x}{1-(x^2-1)}=\tan^{-1}2\Rightarrow\frac{2x}{2-x^2}=2$

$2x^2+2x-4=0$

$x=-2$ or $x=1$

For $x=-2$ both inverse tangents are negative (or the product condition fails), so the sum is not $\tan^{-1}2$; reject it.

$\therefore x=1$

Ans: $x=1$

---

11. Prove that $\tan^{-1}\frac{1}{2}+\tan^{-1}\frac{1}{4}+\tan^{-1}\frac{1}{13}=\frac{\pi}{4}$. [5]

Let $x=\frac{1}{2}$, $y=\frac{1}{4}$, $z=\frac{1}{13}$, all positive.

$S_1=x+y+z=\frac{43}{52},\ S_2=xy+yz+zx=\frac{19}{104}<1,\ S_3=xyz=\frac{1}{104}$

$\tan^{-1}x+\tan^{-1}y+\tan^{-1}z=\tan^{-1}\frac{S_1-S_3}{1-S_2}=\tan^{-1}\frac{\frac{85}{104}}{\frac{85}{104}}$

$=\tan^{-1}1=\frac{\pi}{4}$

Ans: Proved

---

12. Express $\cos^{-1}\left(-\frac{5}{13}\right)$ as an inverse sine. [2]

$x=-\frac{5}{13}<0$, $\sqrt{1-x^2}=\frac{12}{13}$.

$\cos^{-1}x=\pi-\sin^{-1}\sqrt{1-x^2}$

$\cos^{-1}\left(-\frac{5}{13}\right)=\pi-\sin^{-1}\frac{12}{13}$

Ans: $\pi-\sin^{-1}\frac{12}{13}$

---

13. Express $2\tan^{-1}3$ as a single inverse tangent, a single inverse sine and a single inverse cosine. [3]

$x=3>1$.

$2\tan^{-1}x=\pi+\tan^{-1}\frac{2x}{1-x^2}=\pi+\tan^{-1}\left(-\frac{3}{4}\right)=\pi-\tan^{-1}\frac{3}{4}$

$2\tan^{-1}x=\pi-\sin^{-1}\frac{2x}{1+x^2}=\pi-\sin^{-1}\frac{3}{5}$

$2\tan^{-1}x=\cos^{-1}\frac{1-x^2}{1+x^2}=\cos^{-1}\left(-\frac{4}{5}\right)$ (as $x\geq 0$)

Ans: $\pi-\tan^{-1}\frac{3}{4}$, $\pi-\sin^{-1}\frac{3}{5}$, $\cos^{-1}\left(-\frac{4}{5}\right)$

---

14. Write the horizontal asymptotes of the graph of $y = \tan^{-1} x$. [1]

$\tan^{-1} x \to \frac{\pi}{2}$ as $x \to \infty$ and $\to -\frac{\pi}{2}$ as $x \to -\infty$.

Ans: $y = \frac{\pi}{2}$ **and** $y = -\frac{\pi}{2}$

---

15. Find $\cos^{-1}(\cos 5)$. [2]

5 is not in the principal range $[0, \pi]$.

Using the periodic graph of $\cos^{-1}(\cos x)$, shift or reflect it into the range: the value is $-5 + 2\pi$.

Check: $-5 + 2\pi$ is in $[0, \pi]$ and $\cos(-5 + 2\pi) = \cos(5)$.

Ans: $-5 + 2\pi$

---

16. Simplify $\tan^{-1} \frac{2x}{1-x^2}$ for $x > 1$. [3]

Put $x = \tan\theta$, so that the expression inside is a ratio of 2θ.

For $x > 1$ the angle has to be brought back into the principal range.

$\tan^{-1} \frac{2x}{1-x^2} = -\pi + 2\tan^{-1} x$

Ans: $-\pi + 2\tan^{-1} x$

---

17. If $\sin^{-1} \frac{8}{x} + \sin^{-1} \frac{15}{x} = \frac{\pi}{2}$ with $x > 0$, find x. [3]

$\sin^{-1} \frac{8}{x} = \frac{\pi}{2} - \sin^{-1} \frac{15}{x} = \cos^{-1} \frac{15}{x}$.

So $\left(\frac{8}{x}\right)^2 + \left(\frac{15}{x}\right)^2 = 1$, giving $x = \sqrt{64 + 225} = 17$.

Ans: $x = 17$

---

18. Prove that $\tan^{-1} \frac{1}{1+2\cdot3} + \tan^{-1} \frac{1}{1+3\cdot4} + \cdots + \tan^{-1} \frac{1}{1+(n+1)(n+2)} = \tan^{-1} \frac{n}{1+2(n+2)}$. [5]

The kth term is $\tan^{-1} \frac{1}{1+(k+1)(k+2)}$.

$\tan^{-1}(k+2) - \tan^{-1}(k+1) = \tan^{-1} \frac{(k+2)-(k+1)}{1+(k+2)(k+1)} = \tan^{-1} \frac{1}{1+(k+1)(k+2)}$

Adding for $k = 1, 2, \ldots, n$, all middle terms cancel:

$\text{Sum} = \tan^{-1}(n+2) - \tan^{-1} 2$

$= \tan^{-1} \frac{(n+2)-2}{1+2(n+2)} = \tan^{-1} \frac{n}{1+2(n+2)}$

Ans: Proved

---

19. Which substitution removes the square root in $\sqrt{64 + x^2}$? What does the root become? [1]

For $\sqrt{64 + x^2}$ put $x = 8\tan\theta$.

Then the root becomes $8\sec\theta$.

Ans: $x = 8\tan\theta$**; the root becomes** $8\sec\theta$

---

20. Simplify $\sin^{-1} \frac{x}{\sqrt{4+x^2}}$ for $x \in \mathbb{R}$. [2]

Put $x = 2\tan\theta$; then $\sqrt{4 + x^2} = 2\sec\theta$.

$\sin^{-1} \frac{2\tan\theta}{2\sec\theta} = \sin^{-1}(\sin\theta) = \theta = \tan^{-1} \frac{x}{2}$

Ans: $\tan^{-1} \frac{x}{2}$
