EXPRESSWALA ACADEMY

Maths 1A: Inverse Trigonometric Functions

Question Paper, Set 10

Time: 1 hour 30 minutes | Maximum marks: 50

Name: _____ Class: _____ Date: _____

1. Answer all 20 questions. The marks for each question are shown in brackets.
2. Show your working for the 2, 3 and 5 mark questions.
3. The key sheet and worked solutions are at the end of this paper.

1. For what values of x is $\cos^{-1}(5x+2)$ defined? [1 mark]

2. Find the principal value of $\sec^{-1}(-1)$. [1 mark]

3. Evaluate $\tan^{-1}1+\sec^{-1}\frac{2}{\sqrt{3}}$. [2 marks]

4. Find $\cot\left(\cot^{-1}\left(-\frac{7}{3}\right)\right)$. [1 mark]

5. Using the rules for negative arguments, evaluate $\cos^{-1}\left(-\frac{1}{\sqrt{2}}\right)+\sec^{-1}(-1)$. [2 marks]

6. Express $\cot^{-1}\left(-\frac{3}{8}\right)$ in terms of an inverse tangent. [1 mark]

7. Solve $\sin^{-1}x=1\cos^{-1}x$. [2 marks]

8. Prove that $\sin^{-1}\frac{20}{29}-\sin^{-1}\frac{3}{5}=\sin^{-1}\frac{17}{145}$. [5 marks]

9. Find $\tan^{-1}2-\tan^{-1}\frac{1}{3}$. [3 marks]

10. Solve $\tan^{-1}(x+4)+\tan^{-1}(x-4)=\tan^{-1}8$. [5 marks]

11. Prove that $\tan^{-1}\frac{2}{3}+\tan^{-1}\frac{1}{4}+\tan^{-1}\frac{1}{8}=\tan^{-1}\frac{98}{69}$. [5 marks]

12. Express $\tan^{-1}\frac{7}{24}$ as an inverse sine and as an inverse cosine. [2 marks]

13. Express $2\tan^{-1}5$ as a single inverse tangent, a single inverse sine and a single inverse cosine. [3 marks]

14. What is the slope of the graph of $y=\sin^{-1}x$ at the origin? [1 mark]

15. Find $\sin^{-1}\left(\sin\frac{9\pi}{7}\right)$. [2 marks]

16. Simplify $\sin^{-1}(3x-4x^3)$ for $-1\le x\le-\frac{1}{2}$. [3 marks]

17. If $\cos^{-1}x+\cos^{-1}y=\frac{3\pi}{5}$, find $\sin^{-1}x+\sin^{-1}y$. [3 marks]

18. If $\cos^{-1}\frac{x}{5}+\cos^{-1}\frac{y}{6}=\theta$, prove that $36x^2-60xy\cos\theta+25y^2=900\sin^2\theta$. [5 marks]

19. Which substitution removes the square root in $\sqrt{x^2-49}$? What does the root become? [1 mark]

20. Simplify $\tan^{-1}\frac{x}{\sqrt{4-x^2}}$ for $|x| < 2$. [2 marks]

Key sheet

Q	Answer
1	$[-\frac{3}{5}, -\frac{1}{5}]$
2	π
3	$\frac{5\pi}{12}$
4	$-\frac{7}{3}$
5	$\frac{7\pi}{4}$
6	$\pi - \tan^{-1}\frac{8}{3}$
7	$x = \frac{1}{\sqrt{2}}$
8	Proved: $\sin^{-1}\frac{20}{29} - \sin^{-1}\frac{3}{5} = \sin^{-1}\frac{17}{145}$
9	$\frac{\pi}{4}$
10	$x = 4$
11	Proved
12	$\sin^{-1}\frac{7}{25}$, $\cos^{-1}\frac{24}{25}$
13	$\pi - \tan^{-1}\frac{5}{12}$, $\pi - \sin^{-1}\frac{5}{13}$, $\cos^{-1}\left(-\frac{12}{13}\right)$
14	1
15	$-\frac{2\pi}{7}$
16	$-(\pi + 3\sin^{-1}x)$
17	$\frac{2\pi}{5}$
18	Proved
19	$x = 7\sec\theta$; the root becomes $7\lvert\tan\theta\rvert$
20	$\sin^{-1}\frac{x}{2}$

Solutions

1. For what values of x is $\cos^{-1}(5x+2)$ defined? [1]

$-1 \le 5x+2 \le 1$

$-3 \le 5x \le -1$

$-\frac{3}{5} \le x \le -\frac{1}{5}$

Ans: $[-\frac{3}{5}, -\frac{1}{5}]$

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2. Find the principal value of $\sec^{-1}(-1)$. [1]

We need $\theta \in [0, \frac{\pi}{2}) \cup (\frac{\pi}{2}, \pi]$ with $\sec\theta = -1$.

$\therefore \sec^{-1}(-1) = \pi$

Ans: π

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3. Evaluate $\tan^{-1} 1 + \sec^{-1} \frac{2}{\sqrt{3}}$. [2]

$\tan^{-1} 1 = \frac{\pi}{4},\ \sec^{-1} \frac{2}{\sqrt{3}} = \frac{\pi}{6}$

$\frac{\pi}{4} + \frac{\pi}{6} = \frac{5\pi}{12}$

Ans: $\frac{5\pi}{12}$

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4. Find $\cot\left(\cot^{-1}\left(-\frac{7}{3}\right)\right)$. [1]

$\cot(\cot^{-1} x) = x$ for every x in the domain of $\cot^{-1}$.

$-\frac{7}{3}$ is in the domain, so the value is $-\frac{7}{3}$.

Ans: $-\frac{7}{3}$

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5. Using the rules for negative arguments, evaluate $\cos^{-1}\left(-\frac{1}{\sqrt{2}}\right) + \sec^{-1}(-1)$. [2]

$\cos^{-1}\left(-\frac{1}{\sqrt{2}}\right) = \pi - \cos^{-1}\frac{1}{\sqrt{2}} = \pi - \frac{\pi}{4} = \frac{3\pi}{4}$

$\sec^{-1}(-1) = \pi - \sec^{-1} 1 = \pi - 0 = \pi$

$\text{Sum} = \frac{7\pi}{4}$

Ans: $\frac{7\pi}{4}$

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6. Express $\cot^{-1}\left(-\frac{3}{8}\right)$ in terms of an inverse tangent. [1]

$\cot^{-1} x = \pi + \tan^{-1}\frac{1}{x},\ x < 0$

$\cot^{-1}\left(-\frac{3}{8}\right) = \pi + \tan^{-1}\left(-\frac{8}{3}\right) = \pi - \tan^{-1}\frac{8}{3}$

Ans: $\pi - \tan^{-1}\frac{8}{3}$

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7. Solve $\sin^{-1} x = 1\cos^{-1} x$. [2]

Let $\cos^{-1} x = t$; then $\sin^{-1} x = \frac{\pi}{2} - t$.

$\frac{\pi}{2} - t = 1t \Rightarrow t = \frac{\pi}{4}$

$\cos^{-1} x = \frac{\pi}{4} \Rightarrow x = \cos\frac{\pi}{4} = \frac{1}{\sqrt{2}}$

Ans: $x = \frac{1}{\sqrt{2}}$

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8. Prove that $\sin^{-1}\frac{20}{29} - \sin^{-1}\frac{3}{5} = \sin^{-1}\frac{17}{145}$. [5]

Let $x = \frac{20}{29}$, $y = \frac{3}{5}$.

$\sqrt{1-x^2} = \frac{21}{29},\ \sqrt{1-y^2} = \frac{4}{5}$

$x^2 + y^2 = \frac{17569}{21025} \le 1.$

$\sin^{-1}\left(x\sqrt{1-y^2} - y\sqrt{1-x^2}\right) = \sin^{-1}\left(\frac{20}{29}\cdot\frac{4}{5} - \frac{3}{5}\cdot\frac{21}{29}\right) = \sin^{-1}\frac{17}{145}$

Hence $\sin^{-1}\frac{20}{29} - \sin^{-1}\frac{3}{5} = \sin^{-1}\frac{17}{145}$.

Ans: Proved: $\sin^{-1}\frac{20}{29} - \sin^{-1}\frac{3}{5} = \sin^{-1}\frac{17}{145}$

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9. Find $\tan^{-1}2 - \tan^{-1}\frac{1}{3}$. [3]

Here $xy = \frac{2}{3} > -1$.

$\frac{x-y}{1+xy} = \frac{\frac{5}{3}}{\frac{5}{3}} = 1$

$\tan^{-1}2 - \tan^{-1}\frac{1}{3} = \tan^{-1}1 = \frac{\pi}{4}$

Ans: $\frac{\pi}{4}$

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10. Solve $\tan^{-1}(x+4) + \tan^{-1}(x-4) = \tan^{-1}8$. [5]

$\tan^{-1}\frac{2x}{1-(x^2-16)} = \tan^{-1}8 \Rightarrow \frac{2x}{17-x^2} = 8$

$8x^2 + 2x - 136 = 0$

$x = -\frac{17}{4}$ or $x = 4$

For $x = -\frac{17}{4}$ both inverse tangents are negative (or the product condition fails), so the sum is not $\tan^{-1}8$; reject it.

$\therefore x = 4$

Ans: $x = 4$

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11. Prove that $\tan^{-1}\frac{2}{3} + \tan^{-1}\frac{1}{4} + \tan^{-1}\frac{1}{8} = \tan^{-1}\frac{98}{69}$. [5]

Let $x = \frac{2}{3}$, $y = \frac{1}{4}$, $z = \frac{1}{8}$, all positive.

$S_1 = x+y+z = \frac{25}{24},\ S_2 = xy+yz+zx = \frac{9}{32} < 1,\ S_3 = xyz = \frac{1}{48}$

$\tan^{-1}x + \tan^{-1}y + \tan^{-1}z = \tan^{-1}\frac{S_1 - S_3}{1-S_2} = \tan^{-1}\frac{\frac{49}{48}}{\frac{23}{32}}$

$= \tan^{-1}\frac{98}{69}$

Ans: Proved

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12. Express $\tan^{-1}\frac{7}{24}$ as an inverse sine and as an inverse cosine. [2]

$\sqrt{1+x^2} = \frac{25}{24}$

$\tan^{-1}\frac{7}{24} = \sin^{-1}\frac{x}{\sqrt{1+x^2}} = \sin^{-1}\frac{7}{25}$

$\tan^{-1}\frac{7}{24} = \cos^{-1}\frac{1}{\sqrt{1+x^2}} = \cos^{-1}\frac{24}{25}$

Ans: $\sin^{-1}\frac{7}{25}$, $\cos^{-1}\frac{24}{25}$

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13. Express $2\tan^{-1}5$ as a single inverse tangent, a single inverse sine and a single inverse cosine. [3]

$x = 5 > 1.$

$2\tan^{-1}x = \pi + \tan^{-1}\frac{2x}{1-x^2} = \pi + \tan^{-1}\left(-\frac{5}{12}\right) = \pi - \tan^{-1}\frac{5}{12}$

$2\tan^{-1}x = \pi - \sin^{-1}\frac{2x}{1+x^2} = \pi - \sin^{-1}\frac{5}{13}$

$2\tan^{-1}x = \cos^{-1}\frac{1-x^2}{1+x^2} = \cos^{-1}\left(-\frac{12}{13}\right)$ (as $x \ge 0$)

Ans: $\pi - \tan^{-1}\frac{5}{12}$, $\pi - \sin^{-1}\frac{5}{13}$, $\cos^{-1}\left(-\frac{12}{13}\right)$

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14. What is the slope of the graph of $y = \sin^{-1} x$ at the origin? [1]

The graph is the reflection of $y = \sin x$, which has slope 1 at the origin, so its slope there is also 1.

Ans: 1

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15. Find $\sin^{-1}\left(\sin \frac{9\pi}{7}\right)$. [2]

$\frac{9\pi}{7}$ is not in the principal range $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

Using the periodic graph of $\sin^{-1}(\sin x)$, shift or reflect it into the range: the value is $-\frac{2\pi}{7}$.

Check: $-\frac{2\pi}{7}$ is in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ and $\sin(-\frac{2\pi}{7}) = \sin(\frac{9\pi}{7})$.

Ans: $-\frac{2\pi}{7}$

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16. Simplify $\sin^{-1}(3x - 4x^3)$ for $-1 \le x \le -\frac{1}{2}$. [3]

Put $x = \sin\theta$, so that the expression inside is a ratio of 3θ.

For $-1 \le x \le -\frac{1}{2}$ the angle has to be brought back into the principal range.

$\sin^{-1}(3x - 4x^3) = -(\pi + 3\sin^{-1} x)$

Ans: $-(\pi + 3\sin^{-1} x)$

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17. If $\cos^{-1} x + \cos^{-1} y = \frac{3\pi}{5}$, find $\sin^{-1} x + \sin^{-1} y$. [3]

$\sin^{-1} x = \frac{\pi}{2} - \cos^{-1} x,\ \sin^{-1} y = \frac{\pi}{2} - \cos^{-1} y$

$\sin^{-1} x + \sin^{-1} y = \pi - (\cos^{-1} x + \cos^{-1} y) = \pi - \frac{3\pi}{5} = \frac{2\pi}{5}$

Ans: $\frac{2\pi}{5}$

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18. If $\cos^{-1}\frac{x}{5} + \cos^{-1}\frac{y}{6} = \theta$, prove that $36x^2 - 60xy\cos\theta + 25y^2 = 900\sin^2\theta$. [5]

Let $\cos^{-1}\frac{x}{5} = \alpha$, $\cos^{-1}\frac{y}{6} = \beta$, so $\alpha + \beta = \theta$.

Then $\cos\alpha = \frac{x}{5}$, $\sin\alpha = \sqrt{1 - \frac{x^2}{25}}$, and similarly for β.

$\cos\theta = \cos\alpha\cos\beta - \sin\alpha\sin\beta = \frac{xy}{30} - \sqrt{1 - \frac{x^2}{25}}\sqrt{1 - \frac{y^2}{36}}$

$\left(\frac{xy}{30} - \cos\theta\right)^2 = \left(1 - \frac{x^2}{25}\right)\left(1 - \frac{y^2}{36}\right)$

$\frac{x^2y^2}{900} - \frac{2xy}{30}\cos\theta + \cos^2\theta = 1 - \frac{x^2}{25} - \frac{y^2}{36} + \frac{x^2y^2}{900}$

$\frac{x^2}{25} - \frac{2xy}{30}\cos\theta + \frac{y^2}{36} = \sin^2\theta$

Multiplying by 900: $36x^2 - 60xy\cos\theta + 25y^2 = 900\sin^2\theta$.

Ans: Proved

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19. Which substitution removes the square root in $\sqrt{x^2 - 49}$? What does the root become? [1]

For $\sqrt{x^2 - 49}$ put $x = 7\sec\theta$.

Then the root becomes $7|\tan\theta|$.

Ans: $x = 7\sec\theta$**; the root becomes** $7|\tan\theta|$

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20. Simplify $\tan^{-1}\frac{x}{\sqrt{4 - x^2}}$ for $|x| < 2$. [2]

Put $x = 2\sin\theta$; then $\sqrt{4 - x^2} = 2\cos\theta$.

$\tan^{-1}\frac{2\sin\theta}{2\cos\theta} = \tan^{-1}(\tan\theta) = \theta = \sin^{-1}\frac{x}{2}$

Ans: $\sin^{-1}\frac{x}{2}$
