EXPRESSWALA ACADEMY

Maths 1A: Inverse Trigonometric Functions

Question Paper, Set 01

Time: 1 hour 30 minutes | Maximum marks: 50

Name: _____ Class: _____ Date: _____

1. Answer all 20 questions. The marks for each question are shown in brackets.
2. Show your working for the 2, 3 and 5 mark questions.
3. The key sheet and worked solutions are at the end of this paper.

1. Find the domain of $\sin^{-1}(3x - 2)$. [1 mark]

2. Find the principal value of $\sin^{-1}\frac{\sqrt{3}}{2}$. [1 mark]

3. Evaluate $\sin^{-1}\frac{1}{2} + \cos^{-1}\frac{1}{2}$. [2 marks]

4. Find $\sin^{-1}\left(\sin\frac{\pi}{7}\right)$. [1 mark]

5. Using the rules for negative arguments, evaluate $\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right) + \cos^{-1}\left(-\frac{1}{\sqrt{2}}\right)$. [2 marks]

6. Express $\operatorname{cosec}^{-1}\frac{7}{3}$ as an inverse sine. [1 mark]

7. If $\cos^{-1}x = \frac{2\pi}{9}$, find $\sin^{-1}x$. [2 marks]

8. Prove that $\sin^{-1}\frac{4}{5} + \sin^{-1}\frac{7}{25} = \sin^{-1}\frac{117}{125}$. [5 marks]

9. Find $\tan^{-1}\frac{1}{4} + \tan^{-1}\frac{1}{3}$. [3 marks]

10. Solve $\tan^{-1}3x + \tan^{-1}10x = \frac{\pi}{4}$. [5 marks]

11. Prove that $\tan^{-1}\frac{1}{2} + \tan^{-1}\frac{1}{3} + \tan^{-1}\frac{1}{4} = \tan^{-1}\frac{5}{3}$. [5 marks]

12. Express $\sin^{-1}\frac{8}{17}$ as an inverse cosine and as an inverse tangent. [2 marks]

13. Express $2\tan^{-1}\frac{1}{4}$ as a single inverse tangent, a single inverse sine and a single inverse cosine. [3 marks]

14. Is the graph of $y = \cos^{-1}x$ increasing or decreasing on $[-1, 1]$? [1 mark]

15. Find $\sin^{-1}(\sin 4)$. [2 marks]

16. Find the value of $\sin^{-1}\frac{2x}{1+x^2}$ when $x = \frac{1}{\sqrt{3}}$. [3 marks]

17. If $\tan^{-1}\frac{3}{x} + \tan^{-1}\frac{12}{x} = \frac{\pi}{2}$ with $x > 0$, find x. [3 marks]

18. If $\cos^{-1}\frac{x}{2} + \cos^{-1}\frac{y}{3} = \theta$, prove that $9x^2 - 12xy\cos\theta + 4y^2 = 36\sin^2\theta$. [5 marks]

19. Which substitution removes the square root in $\sqrt{9 - x^2}$? What does the root become? [1 mark]

20. Simplify $\tan^{-1}\frac{x}{\sqrt{9-x^2}}$ for $|x| < 3$. [2 marks]

Key sheet

Q	Answer
1	$\left[\frac{1}{3}, 1\right]$
2	$\frac{\pi}{3}$
3	$\frac{\pi}{2}$
4	$\frac{\pi}{7}$
5	$\frac{5\pi}{12}$
6	$\sin^{-1}\frac{3}{7}$
7	$\frac{5\pi}{18}$
8	Proved: $\sin^{-1}\frac{4}{5} + \sin^{-1}\frac{7}{25} = \sin^{-1}\frac{117}{125}$
9	$\tan^{-1}\frac{7}{11}$
10	$x = \frac{1}{15}$
11	Proved
12	$\cos^{-1}\frac{15}{17}$, $\tan^{-1}\frac{8}{15}$
13	$\tan^{-1}\frac{8}{15}$, $\sin^{-1}\frac{8}{17}$, $\cos^{-1}\frac{15}{17}$
14	Decreasing
15	$-4 + \pi$
16	$\frac{\pi}{3}$
17	$x = 6$
18	Proved
19	$x = 3\sin\theta$; the root becomes $3\cos\theta$
20	$\sin^{-1}\frac{x}{3}$

Solutions

1. Find the domain of $\sin^{-1}(3x-2)$. [1]

$-1 \le 3x-2 \le 1$

$1 \le 3x \le 3$

$\frac{1}{3} \le x \le 1$

Ans: $[\frac{1}{3}, 1]$

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2. Find the principal value of $\sin^{-1}\frac{\sqrt{3}}{2}$. [1]

We need $\theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ with $\sin\theta = \frac{\sqrt{3}}{2}$.

$\therefore \sin^{-1}\frac{\sqrt{3}}{2} = \frac{\pi}{3}$

Ans: $\frac{\pi}{3}$

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3. Evaluate $\sin^{-1}\frac{1}{2} + \cos^{-1}\frac{1}{2}$. [2]

$\sin^{-1}\frac{1}{2} = \frac{\pi}{6},\ \cos^{-1}\frac{1}{2} = \frac{\pi}{3}$

$\frac{\pi}{6} + \frac{\pi}{3} = \frac{\pi}{2}$

Ans: $\frac{\pi}{2}$

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4. Find $\sin^{-1}\left(\sin\frac{\pi}{7}\right)$. [1]

$\frac{\pi}{7}$ lies in the principal range $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

So $\sin^{-1}(\sin\theta) = \theta$ gives $\frac{\pi}{7}$.

Ans: $\frac{\pi}{7}$

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5. Using the rules for negative arguments, evaluate $\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right) + \cos^{-1}\left(-\frac{1}{\sqrt{2}}\right)$. [2]

$\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right) = -\sin^{-1}\frac{\sqrt{3}}{2} = -\frac{\pi}{3}$

$\cos^{-1}\left(-\frac{1}{\sqrt{2}}\right) = \pi - \cos^{-1}\frac{1}{\sqrt{2}} = \pi - \frac{\pi}{4} = \frac{3\pi}{4}$

$\text{Sum} = \frac{5\pi}{12}$

Ans: $\frac{5\pi}{12}$

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6. Express $\text{cosec}^{-1}\frac{7}{3}$ as an inverse sine. [1]

$\text{cosec}^{-1}x = \sin^{-1}\frac{1}{x},\ |x| \ge 1$

$\text{cosec}^{-1}\frac{7}{3} = \sin^{-1}\frac{3}{7}$

Ans: $\sin^{-1}\frac{3}{7}$

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7. If $\cos^{-1}x = \frac{2\pi}{9}$, find $\sin^{-1}x$. [2]

$\cos^{-1}x + \sin^{-1}x = \frac{\pi}{2}$

$\sin^{-1}x = \frac{\pi}{2} - \frac{2\pi}{9} = \frac{5\pi}{18}$

Ans: $\frac{5\pi}{18}$

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8. Prove that $\sin^{-1}\frac{4}{5}+\sin^{-1}\frac{7}{25}=\sin^{-1}\frac{117}{125}$. [5]

Let $x=\frac{4}{5}$, $y=\frac{7}{25}$.

$\sqrt{1-x^2}=\frac{3}{5},\ \sqrt{1-y^2}=\frac{24}{25}$

$x^2+y^2=\frac{449}{625}\le 1$.

$\sin^{-1}\left(x\sqrt{1-y^2}+y\sqrt{1-x^2}\right)=\sin^{-1}\left(\frac{4}{5}\cdot\frac{24}{25}+\frac{7}{25}\cdot\frac{3}{5}\right)=\sin^{-1}\frac{117}{125}$

Hence $\sin^{-1}\frac{4}{5}+\sin^{-1}\frac{7}{25}=\sin^{-1}\frac{117}{125}$.

Ans: Proved: $\sin^{-1}\frac{4}{5}+\sin^{-1}\frac{7}{25}=\sin^{-1}\frac{117}{125}$

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9. Find $\tan^{-1}\frac{1}{4}+\tan^{-1}\frac{1}{3}$. [3]

Here $xy=\frac{1}{12}<1$.

$\frac{x+y}{1-xy}=\frac{\frac{7}{12}}{\frac{11}{12}}=\frac{7}{11}$

$\tan^{-1}\frac{1}{4}+\tan^{-1}\frac{1}{3}=\tan^{-1}\frac{7}{11}$

Ans: $\tan^{-1}\frac{7}{11}$

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10. Solve $\tan^{-1}3x+\tan^{-1}10x=\frac{\pi}{4}$. [5]

$\tan^{-1}\frac{13x}{1-30x^2}=\frac{\pi}{4}\Rightarrow\frac{13x}{1-30x^2}=1$

$30x^2+13x-1=0$

$x=-\frac{1}{2}$ or $x=\frac{1}{15}$

For $x=-\frac{1}{2}$ both inverse tangents are negative, so the left side is negative and cannot equal $\frac{\pi}{4}$; reject it.

$\therefore x=\frac{1}{15}$

Ans: $x=\frac{1}{15}$

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11. Prove that $\tan^{-1}\frac{1}{2}+\tan^{-1}\frac{1}{3}+\tan^{-1}\frac{1}{4}=\tan^{-1}\frac{5}{3}$. [5]

Let $x=\frac{1}{2}$, $y=\frac{1}{3}$, $z=\frac{1}{4}$, all positive.

$S_1=x+y+z=\frac{13}{12},\ S_2=xy+yz+zx=\frac{3}{8}<1,\ S_3=xyz=\frac{1}{24}$

$\tan^{-1}x+\tan^{-1}y+\tan^{-1}z=\tan^{-1}\frac{S_1-S_3}{1-S_2}=\tan^{-1}\frac{\frac{25}{24}}{\frac{5}{8}}$

$=\tan^{-1}\frac{5}{3}$

Ans: Proved

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12. Express $\sin^{-1}\frac{8}{17}$ as an inverse cosine and as an inverse tangent. [2]

$x=\frac{8}{17}>0$, $\sqrt{1-x^2}=\frac{15}{17}$.

$\sin^{-1}\frac{8}{17}=\cos^{-1}\sqrt{1-x^2}=\cos^{-1}\frac{15}{17}$

$\sin^{-1}\frac{8}{17}=\tan^{-1}\frac{x}{\sqrt{1-x^2}}=\tan^{-1}\frac{8}{15}$

Ans: $\cos^{-1}\frac{15}{17}$, $\tan^{-1}\frac{8}{15}$

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13. Express $2\tan^{-1}\frac{1}{4}$ as a single inverse tangent, a single inverse sine and a single inverse cosine. [3]

$0<x=\frac{1}{4}<1$.

$2\tan^{-1}x=\tan^{-1}\frac{2x}{1-x^2}=\tan^{-1}\frac{\frac{1}{2}}{\frac{15}{16}}=\tan^{-1}\frac{8}{15}$

$2\tan^{-1}x=\sin^{-1}\frac{2x}{1+x^2}=\sin^{-1}\frac{\frac{1}{2}}{\frac{17}{16}}=\sin^{-1}\frac{8}{17}$

$2\tan^{-1}x=\cos^{-1}\frac{1-x^2}{1+x^2}=\cos^{-1}\frac{\frac{15}{16}}{\frac{17}{16}}=\cos^{-1}\frac{15}{17}$

Ans: $\tan^{-1}\frac{8}{15}$, $\sin^{-1}\frac{8}{17}$, $\cos^{-1}\frac{15}{17}$

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14. Is the graph of $y = \cos^{-1} x$ increasing or decreasing on $[-1, 1]$? [1]

It runs from $(-1, \pi)$ through $\left(0, \frac{\pi}{2}\right)$ to $(1, 0)$, so it is decreasing.

Ans: Decreasing

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15. Find $\sin^{-1}(\sin 4)$. [2]

4 is not in the principal range $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

Using the periodic graph of $\sin^{-1}(\sin x)$, shift or reflect it into the range: the value is $-4 + \pi$.

Check: $-4 + \pi$ is in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ and $\sin(-4 + \pi) = \sin(4)$.

Ans: $-4 + \pi$

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16. Find the value of $\sin^{-1} \frac{2x}{1+x^2}$ when $x = \frac{1}{\sqrt{3}}$. [3]

$x = \frac{1}{\sqrt{3}}$ lies in $-1 < x < 1$, where $\sin^{-1} \frac{2x}{1+x^2} = 2 \tan^{-1} x$.

$= \frac{\pi}{3}$

Ans: $\frac{\pi}{3}$

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17. If $\tan^{-1} \frac{3}{x} + \tan^{-1} \frac{12}{x} = \frac{\pi}{2}$ with $x > 0$, find x. [3]

Two inverse tangents add to $\frac{\pi}{2}$ when the product of the values is 1.

$\frac{3}{x} \cdot \frac{12}{x} = 1 \Rightarrow x^2 = 36 \Rightarrow x = 6$

Ans: $x = 6$

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18. If $\cos^{-1} \frac{x}{2} + \cos^{-1} \frac{y}{3} = \theta$, prove that $9x^2 - 12xy \cos\theta + 4y^2 = 36 \sin^2 \theta$. [5]

Let $\cos^{-1} \frac{x}{2} = \alpha$, $\cos^{-1} \frac{y}{3} = \beta$, so $\alpha + \beta = \theta$.

Then $\cos\alpha = \frac{x}{2}$, $\sin\alpha = \sqrt{1 - \frac{x^2}{4}}$, and similarly for β.

$\cos\theta = \cos\alpha\cos\beta - \sin\alpha\sin\beta = \frac{xy}{6} - \sqrt{1 - \frac{x^2}{4}}\sqrt{1 - \frac{y^2}{9}}$

$\left(\frac{xy}{6} - \cos\theta\right)^2 = \left(1 - \frac{x^2}{4}\right)\left(1 - \frac{y^2}{9}\right)$

$\frac{x^2y^2}{36} - \frac{2xy}{6}\cos\theta + \cos^2\theta = 1 - \frac{x^2}{4} - \frac{y^2}{9} + \frac{x^2y^2}{36}$

$\frac{x^2}{4} - \frac{2xy}{6}\cos\theta + \frac{y^2}{9} = \sin^2\theta$

Multiplying by 36: $9x^2 - 12xy\cos\theta + 4y^2 = 36\sin^2\theta$.

Ans: Proved

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19. Which substitution removes the square root in $\sqrt{9 - x^2}$? What does the root become? [1]

For $\sqrt{9 - x^2}$ put $x = 3\sin\theta$.

Then the root becomes $3\cos\theta$.

Ans: $x = 3\sin\theta$**; the root becomes** $3\cos\theta$

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20. Simplify $\tan^{-1} \frac{x}{\sqrt{9-x^2}}$ for $|x| < 3$. [2]

Put $x = 3\sin\theta$; then $\sqrt{9 - x^2} = 3\cos\theta$.

$\tan^{-1} \frac{3\sin\theta}{3\cos\theta} = \tan^{-1}(\tan\theta) = \theta = \sin^{-1} \frac{x}{3}$

Ans: $\sin^{-1} \frac{x}{3}$
