EXPRESSWALA ACADEMY

Maths 1A: Inverse of a Matrix

Question Paper, Set 04

Time: 1 hour 30 minutes | Maximum marks: 50

Name: _____ Class: _____ Date: _____

1. Answer all 20 questions. The marks for each question are shown in brackets.
2. Show your working for the 2, 3 and 5 mark questions.
3. The key sheet and worked solutions are at the end of this paper.

1. Find the cofactor of the element in row 2, column 3 of $A = \begin{bmatrix} 2 & 1 & 0 \\ -4 & -2 & -2 \\ 5 & 6 & 1 \end{bmatrix}$. [1 mark]

2. Find the adjoint of $A = \begin{bmatrix} 1 & -2 & -2 \\ 2 & 1 & 1 \\ -2 & -2 & -3 \end{bmatrix}$. [3 marks]

3. If A is invertible and $|A^{-1}| = \frac{1}{5}$, find $|A|$. [1 mark]

4. If $A^{-1} = \begin{bmatrix} 0 & -1 \\ -1 & 2 \end{bmatrix}$ and $B^{-1} = \begin{bmatrix} 1 & 0 \\ -3 & -1 \end{bmatrix}$, find $(AB)^{-1}$. [2 marks]

5. Find the values of x for which $\begin{bmatrix} x & -27 \\ -3 & x \end{bmatrix}$ is without an inverse. [2 marks]

6. Write down the inverse of $\begin{bmatrix} -4 & -3 \\ -3 & -2 \end{bmatrix}$. [1 mark]

7. Find the inverse of $A = \begin{bmatrix} 2 & 9 \\ -1 & 1 \end{bmatrix}$. [2 marks]

8. Find the inverse of $A = \begin{bmatrix} 1 & -2 & 0 \\ -2 & 3 & 4 \\ 0 & 0 & 3 \end{bmatrix}$. [5 marks]

9. Find the inverse of $\begin{bmatrix} 5 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & 2 \end{bmatrix}$. [1 mark]

10. Verify that $A = \frac{1}{17}\begin{bmatrix} 8 & 15 \\ -15 & 8 \end{bmatrix}$ satisfies $AA^T = I$ and write down A^{-1}. [3 marks]

11. If $A = \begin{bmatrix} -5 & 3 \\ 8 & -3 \end{bmatrix}$, find $\operatorname{Adj} A$ and show that $(\operatorname{Adj} A)A = |A|\,I$. [2 marks]

12. Find the adjoint of $A = \begin{bmatrix} 3 & -2 & 1 \\ 4 & -2 & -1 \\ 0 & 0 & -2 \end{bmatrix}$ and show that $A(\operatorname{Adj} A) = |A|\,I$. [5 marks]

13. If A is a square matrix of order 3 and $|A| = 2$, find $|\operatorname{Adj} A|$. [1 mark]

14. For $A = \begin{bmatrix} -1 & 1 & 1 \\ -2 & 3 & -2 \\ -2 & 3 & 0 \end{bmatrix}$, find $\operatorname{Adj} A$ and verify that $|\operatorname{Adj} A| = |A|^2$. [5 marks]

15. A is a non-singular matrix of order 3 with $|A| = 5$. Express $\operatorname{Adj}(\operatorname{Adj} A)$ in terms of A. [1 mark]

16. If $A = \begin{bmatrix} -4 & 7 \\ 2 & -2 \end{bmatrix}$, find $(\operatorname{Adj} A)^{-1}$. [2 marks]

17. Find S_1, S_2, S_3 for $A = \begin{bmatrix} -2 & 2 & -1 \\ -1 & 0 & 2 \\ 2 & 4 & 0 \end{bmatrix}$ and write its characteristic equation. [3 marks]

18. If a matrix A satisfies $A^2 - 7A + 2I = O$, express A^{-1} in terms of A and I. [2 marks]

19. If $A = \begin{bmatrix} 7 & -2 \\ -3 & 2 \end{bmatrix}$, show that $A^2 - 9A + 8I = O$. Use it to find A^{-1}. [3 marks]

20. Using the characteristic equation, find the inverse of $A = \begin{bmatrix} -1 & 3 & 2 \\ 3 & 1 & 3 \\ 1 & -1 & -1 \end{bmatrix}$. [5 marks]

Key sheet

Q	Answer
1	-7
2	$\operatorname{Adj} A = \begin{bmatrix} -1 & -2 & 0 \\ 4 & -7 & -5 \\ -2 & 6 & 5 \end{bmatrix}$
3	5
4	$(AB)^{-1} = \begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix}$
5	$x = \pm 9$
6	$\begin{bmatrix} 2 & -3 \\ -3 & 4 \end{bmatrix}$
7	$A^{-1} = \frac{1}{11}\begin{bmatrix} 1 & -9 \\ 1 & 2 \end{bmatrix}$
8	$A^{-1} = -\frac{1}{3}\begin{bmatrix} 9 & 6 & -8 \\ 6 & 3 & -4 \\ 0 & 0 & -1 \end{bmatrix}$
9	$\begin{bmatrix} \frac{1}{5} & 0 & 0 \\ 0 & \frac{1}{7} & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix}$
10	$A^{-1} = A^{T} = \frac{1}{17}\begin{bmatrix} 8 & -15 \\ 15 & 8 \end{bmatrix}$
11	$\operatorname{Adj} A = \begin{bmatrix} -3 & -3 \\ -8 & -5 \end{bmatrix}$; the product is $-9I$
12	$\operatorname{Adj} A = \begin{bmatrix} 4 & -4 & 4 \\ 8 & -6 & 7 \\ 0 & 0 & 2 \end{bmatrix}$; both products equal $-4I$
13	4
14	$\lvert A\rvert = -2,\ \lvert\operatorname{Adj} A\rvert = 4 = \lvert A\rvert^2$
15	$5A$
16	$\frac{1}{6}\begin{bmatrix} 4 & -7 \\ -2 & 2 \end{bmatrix}$
17	$\lambda^3 + 2\lambda^2 - 4\lambda - 28 = 0$
18	$A^{-1} = \frac{1}{2}(7I - A)$
19	$A^{-1} = \frac{1}{8}\begin{bmatrix} 2 & 2 \\ 3 & 7 \end{bmatrix}$
20	$A^{-1} = \frac{1}{8}\begin{bmatrix} 2 & 1 & 7 \\ 6 & -1 & 9 \\ -4 & 2 & -10 \end{bmatrix}$

Solutions

1. Find the cofactor of the element in row 2, column 3 of $A = \begin{bmatrix} 2 & 1 & 0 \\ -4 & -2 & -2 \\ 5 & 6 & 1 \end{bmatrix}$. [1]

$\text{Minor} = \begin{vmatrix} 2 & 1 \\ 5 & 6 \end{vmatrix} = 7$

$A_{23} = (-1)^{2+3}(7) = -7$

Ans: -7

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2. Find the adjoint of $A = \begin{bmatrix} 1 & -2 & -2 \\ 2 & 1 & 1 \\ -2 & -2 & -3 \end{bmatrix}$. [3]

The cofactors are

$A_{11} = -1,\ A_{12} = 4,\ A_{13} = -2$

$A_{21} = -2,\ A_{22} = -7,\ A_{23} = 6$

$A_{31} = 0,\ A_{32} = -5,\ A_{33} = 5$

The cofactors of each row form a column of the adjoint:

$\text{Adj}\, A = \begin{bmatrix} -1 & -2 & 0 \\ 4 & -7 & -5 \\ -2 & 6 & 5 \end{bmatrix}$

Ans: $\text{Adj}\, A = \begin{bmatrix} -1 & -2 & 0 \\ 4 & -7 & -5 \\ -2 & 6 & 5 \end{bmatrix}$

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3. If A is invertible and $|A^{-1}| = \frac{1}{5}$, find $|A|$. [1]

$|A|\,|A^{-1}| = 1$

$|A| = 5$

Ans: 5

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4. If $A^{-1} = \begin{bmatrix} 0 & -1 \\ -1 & 2 \end{bmatrix}$ and $B^{-1} = \begin{bmatrix} 1 & 0 \\ -3 & -1 \end{bmatrix}$, find $(AB)^{-1}$. [2]

$(AB)^{-1} = B^{-1}A^{-1}$

$= \begin{bmatrix} 1 & 0 \\ -3 & -1 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ -1 & 2 \end{bmatrix}$

$= \begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix}$

Ans: $(AB)^{-1} = \begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix}$

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5. Find the values of x for which $\begin{bmatrix} x & -27 \\ -3 & x \end{bmatrix}$ is without an inverse. [2]

$|A| = x^2 - 81 = 0$

$x = \pm 9$

Ans: $x = \pm 9$

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6. Write down the inverse of $\begin{bmatrix} -4 & -3 \\ -3 & -2 \end{bmatrix}$. [1]

$|A| = -1$

Swap the diagonal entries, change the signs of the other two, divide by -1.

$A^{-1} = \begin{bmatrix} 2 & -3 \\ -3 & 4 \end{bmatrix}$

Ans: $\begin{bmatrix} 2 & -3 \\ -3 & 4 \end{bmatrix}$

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7. Find the inverse of $A = \begin{bmatrix} 2 & 9 \\ -1 & 1 \end{bmatrix}$. [2]

$|A| = (2)(1) - (9)(-1) = 11 \neq 0$

$\text{Adj}\, A = \begin{bmatrix} 1 & -9 \\ 1 & 2 \end{bmatrix}$

$A^{-1} = \frac{1}{11}\begin{bmatrix} 1 & -9 \\ 1 & 2 \end{bmatrix}$

Ans: $A^{-1} = \frac{1}{11}\begin{bmatrix} 1 & -9 \\ 1 & 2 \end{bmatrix}$

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8. Find the inverse of $A = \begin{bmatrix} 1 & -2 & 0 \\ -2 & 3 & 4 \\ 0 & 0 & 3 \end{bmatrix}$. [5]

Cofactors:

$A_{11} = 9,\ A_{12} = 6,\ A_{13} = 0$

$A_{21} = 6,\ A_{22} = 3,\ A_{23} = 0$

$A_{31} = -8,\ A_{32} = -4,\ A_{33} = -1$

$|A| = 1(9) + (-2)(6) + 0(0) = -3 \neq 0$

$\text{Adj}\, A = \begin{bmatrix} 9 & 6 & -8 \\ 6 & 3 & -4 \\ 0 & 0 & -1 \end{bmatrix}$

$A^{-1} = \frac{\text{Adj}\, A}{|A|} = -\frac{1}{3}\begin{bmatrix} 9 & 6 & -8 \\ 6 & 3 & -4 \\ 0 & 0 & -1 \end{bmatrix}$

Ans: $A^{-1} = -\frac{1}{3}\begin{bmatrix} 9 & 6 & -8 \\ 6 & 3 & -4 \\ 0 & 0 & -1 \end{bmatrix}$

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9. Find the inverse of $\begin{bmatrix} 5 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & 2 \end{bmatrix}$. [1]

$\text{diag}(a, b, c)^{-1} = \text{diag}\left(\frac{1}{a}, \frac{1}{b}, \frac{1}{c}\right)$

$= \begin{bmatrix} \frac{1}{5} & 0 & 0 \\ 0 & \frac{1}{7} & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix}$

Ans: $\begin{bmatrix} \frac{1}{5} & 0 & 0 \\ 0 & \frac{1}{7} & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix}$

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10. Verify that $A = \frac{1}{17}\begin{bmatrix} 8 & 15 \\ -15 & 8 \end{bmatrix}$ satisfies $AA^T = I$ and write down A^{-1}. [3]

$A^T = \frac{1}{17}\begin{bmatrix} 8 & -15 \\ 15 & 8 \end{bmatrix}$

$AA^T = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$

So A is orthogonal, and for an orthogonal matrix $A^{-1} = A^T$.

$A^{-1} = \frac{1}{17}\begin{bmatrix} 8 & -15 \\ 15 & 8 \end{bmatrix}$

Ans: $A^{-1} = A^T = \frac{1}{17}\begin{bmatrix} 8 & -15 \\ 15 & 8 \end{bmatrix}$

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11. If $A = \begin{bmatrix} -5 & 3 \\ 8 & -3 \end{bmatrix}$, find $\operatorname{Adj} A$ and show that $(\operatorname{Adj} A)A = |A|\, I$. [2]

$\operatorname{Adj} A = \begin{bmatrix} -3 & -3 \\ -8 & -5 \end{bmatrix}$

$|A| = -9$

$A(\operatorname{Adj} A) = \begin{bmatrix} -9 & 0 \\ 0 & -9 \end{bmatrix} = -9I$

Ans: $\operatorname{Adj} A = \begin{bmatrix} -3 & -3 \\ -8 & -5 \end{bmatrix}$; **the product is** $-9I$

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12. Find the adjoint of $A = \begin{bmatrix} 3 & -2 & 1 \\ 4 & -2 & -1 \\ 0 & 0 & -2 \end{bmatrix}$ and show that $A(\operatorname{Adj} A) = |A|\, I$. [5]

Cofactors:

$A_{11} = 4,\ A_{12} = 8,\ A_{13} = 0$

$A_{21} = -4,\ A_{22} = -6,\ A_{23} = 0$

$A_{31} = 4,\ A_{32} = 7,\ A_{33} = 2$

$\operatorname{Adj} A = \begin{bmatrix} 4 & -4 & 4 \\ 8 & -6 & 7 \\ 0 & 0 & 2 \end{bmatrix}$

$|A| = 3(4) + (-2)(8) + 1(0) = -4$

$A(\operatorname{Adj} A) = \begin{bmatrix} -4 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & -4 \end{bmatrix} = -4I$

$(\operatorname{Adj} A)A = \begin{bmatrix} -4 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & -4 \end{bmatrix} = -4I$

Ans: $\operatorname{Adj} A = \begin{bmatrix} 4 & -4 & 4 \\ 8 & -6 & 7 \\ 0 & 0 & 2 \end{bmatrix}$; **both products equal** $-4I$

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13. If A is a square matrix of order 3 and $|A| = 2$, find $|\operatorname{Adj} A|$. [1]

$|\operatorname{Adj} A| = |A|^{n-1}$

$= (2)^2 = 4$

Ans: 4

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14. For $A = \begin{bmatrix} -1 & 1 & 1 \\ -2 & 3 & -2 \\ -2 & 3 & 0 \end{bmatrix}$, find $\text{Adj } A$ and verify that $|\text{Adj } A| = |A|^2$. [5]

Cofactors:

$A_{11} = 6,\ A_{12} = 4,\ A_{13} = 0$

$A_{21} = 3,\ A_{22} = 2,\ A_{23} = 1$

$A_{31} = -5,\ A_{32} = -4,\ A_{33} = -1$

$\text{Adj } A = \begin{bmatrix} 6 & 3 & -5 \\ 4 & 2 & -4 \\ 0 & 1 & -1 \end{bmatrix}$

$|A| = -1(6) + 1(4) + 1(0) = -2$

$|\text{Adj } A| = \begin{vmatrix} 6 & 3 & -5 \\ 4 & 2 & -4 \\ 0 & 1 & -1 \end{vmatrix} = 4$

$|A|^2 = (-2)^2 = 4$

So $|\text{Adj } A| = |A|^2$.

Ans: $|A| = -2,\ |\text{Adj } A| = 4 = |A|^2$

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15. A is a non-singular matrix of order 3 with $|A| = 5$. Express $\text{Adj}(\text{Adj } A)$ in terms of A. [1]

$\text{Adj}(\text{Adj } A) = |A|^{n-2} A$

$= (5)^1 A = 5A$

Ans: $5A$

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16. If $A = \begin{bmatrix} -4 & 7 \\ 2 & -2 \end{bmatrix}$, find $(\text{Adj } A)^{-1}$. [2]

$(\text{Adj } A)^{-1} = \frac{1}{|A|} A$

$|A| = -6$

$(\text{Adj } A)^{-1} = \frac{1}{6} \begin{bmatrix} 4 & -7 \\ -2 & 2 \end{bmatrix}$

Ans: $\frac{1}{6} \begin{bmatrix} 4 & -7 \\ -2 & 2 \end{bmatrix}$

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17. Find S_1, S_2, S_3 for $A = \begin{bmatrix} -2 & 2 & -1 \\ -1 & 0 & 2 \\ 2 & 4 & 0 \end{bmatrix}$ and write its characteristic equation. [3]

$S_1 = -2 + 0 + 0 = -2$

$S_2 = \begin{vmatrix} -2 & 2 \\ -1 & 0 \end{vmatrix} + \begin{vmatrix} 0 & 2 \\ 4 & 0 \end{vmatrix} + \begin{vmatrix} -2 & -1 \\ 2 & 0 \end{vmatrix} = -4$

$S_3 = |A| = 28$

$\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$

$\lambda^3 + 2\lambda^2 - 4\lambda - 28 = 0$

Ans: $\lambda^3 + 2\lambda^2 - 4\lambda - 28 = 0$

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18. If a matrix A satisfies $A^2 - 7A + 2I = O$, express A^{-1} in terms of A and I. [2]

Multiply by A^{-1}: $A - 7I + 2A^{-1} = O$

$A^{-1} = \frac{1}{2}(7I - A)$

Ans: $A^{-1} = \frac{1}{2}(7I - A)$

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19. If $A = \begin{bmatrix} 7 & -2 \\ -3 & 2 \end{bmatrix}$, show that $A^2 - 9A + 8I = O$. Use it to find A^{-1}. [3]

$A^2 = \begin{bmatrix} 55 & -18 \\ -27 & 10 \end{bmatrix}$

$A^2 - 9A + 8I = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

Multiply by A^{-1}: $A - 9I + 8A^{-1} = O$

$A^{-1} = \frac{1}{8}(9I - A) = \frac{1}{8}\begin{bmatrix} 2 & 2 \\ 3 & 7 \end{bmatrix}$

Ans: $A^{-1} = \frac{1}{8}\begin{bmatrix} 2 & 2 \\ 3 & 7 \end{bmatrix}$

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20. Using the characteristic equation, find the inverse of $A = \begin{bmatrix} -1 & 3 & 2 \\ 3 & 1 & 3 \\ 1 & -1 & -1 \end{bmatrix}$. [5]

$S_1 = -1,\ S_2 = \begin{vmatrix} -1 & 3 \\ 3 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 3 \\ -1 & -1 \end{vmatrix} + \begin{vmatrix} -1 & 2 \\ 1 & -1 \end{vmatrix} = -9,\ S_3 = |A| = 8$

$\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0:\ \lambda^3 + \lambda^2 - 9\lambda - 8 = 0$

$A^3 + A^2 - 9A - 8I = O$

$A^{-1} = \frac{1}{S_3}(A^2 - S_1 A + S_2 I) = \frac{1}{8}(A^2 + A - 9I)$

$A^2 = \begin{bmatrix} 12 & -2 & 5 \\ 3 & 7 & 6 \\ -5 & 3 & 0 \end{bmatrix}$

$A^2 + A - 9I = \begin{bmatrix} 2 & 1 & 7 \\ 6 & -1 & 9 \\ -4 & 2 & -10 \end{bmatrix}$

$A^{-1} = \frac{1}{8}\begin{bmatrix} 2 & 1 & 7 \\ 6 & -1 & 9 \\ -4 & 2 & -10 \end{bmatrix}$

Ans: $A^{-1} = \frac{1}{8}\begin{bmatrix} 2 & 1 & 7 \\ 6 & -1 & 9 \\ -4 & 2 & -10 \end{bmatrix}$