EXPRESSWALA ACADEMY

Maths 1A: Inverse of a Matrix

Question Paper, Set 03

Time: 1 hour 30 minutes | Maximum marks: 50

Name: _____ Class: _____ Date: _____

1. Answer all 20 questions. The marks for each question are shown in brackets.
2. Show your working for the 2, 3 and 5 mark questions.
3. The key sheet and worked solutions are at the end of this paper.

1. If $A = \begin{bmatrix} -4 & 2 & 0 \\ 0 & 0 & -1 \\ 1 & -2 & 4 \end{bmatrix}$, find the cofactor A_{12}. [1 mark]

2. If $A = \begin{bmatrix} 1 & 2 & 3 \\ -2 & -2 & -1 \\ 2 & 0 & 4 \end{bmatrix}$, find $\operatorname{Adj} A$. [3 marks]

3. A is a 3×3 matrix with $|A| = -12$. Find $|A^{-1}|$. [1 mark]

4. If $A^{-1} = \begin{bmatrix} 3 & 5 \\ 1 & 2 \end{bmatrix}$ and $B^{-1} = \begin{bmatrix} 3 & -1 \\ 2 & -1 \end{bmatrix}$, find $(A^T B)^{-1}$. [2 marks]

5. Find the values of x for which $\begin{bmatrix} x & 18 \\ 2 & x \end{bmatrix}$ is singular. [2 marks]

6. Find A^{-1} if $A = \begin{bmatrix} 2 & 5 \\ -1 & -3 \end{bmatrix}$. [1 mark]

7. If $A = \begin{bmatrix} -3 & -4 \\ 1 & -2 \end{bmatrix}$, find A^{-1}. [2 marks]

8. Find the inverse of $A = \begin{bmatrix} -1 & 0 & 2 \\ -2 & 0 & 3 \\ -1 & 1 & 0 \end{bmatrix}$. [5 marks]

9. Write the inverse of the diagonal matrix $\begin{bmatrix} 5 & 0 & 0 \\ 0 & \frac{2}{3} & 0 \\ 0 & 0 & -2 \end{bmatrix}$. [1 mark]

10. Show that $A = \frac{1}{25}\begin{bmatrix} 24 & 7 \\ -7 & 24 \end{bmatrix}$ is orthogonal and hence find A^{-1}. [3 marks]

11. If $A = \begin{bmatrix} 3 & -4 \\ 6 & -4 \end{bmatrix}$, find $\operatorname{Adj} A$ and show that $(\operatorname{Adj} A)A = |A|\, I$. [2 marks]

12. Find the adjoint of $A = \begin{bmatrix} -1 & 1 & 2 \\ -2 & 2 & -2 \\ -2 & 3 & -1 \end{bmatrix}$ and show that $A(\operatorname{Adj} A) = |A|\, I$. [5 marks]

13. A is a 4×4 matrix with $|A| = 4$. Find $|\operatorname{Adj} A|$. [1 mark]

14. If $A = \begin{bmatrix} -2 & 1 & 1 \\ 2 & 2 & 2 \\ 3 & 0 & -1 \end{bmatrix}$, find $|A|$ and $\operatorname{Adj} A$, and show that $|\operatorname{Adj} A| = |A|^2$. [5 marks]

15. If A is a square matrix of order 4, express $\operatorname{Adj}(5A)$ in terms of $\operatorname{Adj} A$. [1 mark]

16. If $A = \begin{bmatrix} -1 & -1 & 1 \\ 2 & 1 & 2 \\ -2 & -1 & 1 \end{bmatrix}$, find $\operatorname{Adj}(\operatorname{Adj} A)$ without finding $\operatorname{Adj} A$. [2 marks]

17. Find the characteristic equation of $A = \begin{bmatrix} 4 & 2 & 2 \\ 0 & 0 & -1 \\ 2 & 2 & 2 \end{bmatrix}$. [3 marks]

18. If a matrix A satisfies $A^2 - 5A - 3I = O$, express A^{-1} in terms of A and I. [2 marks]

19. If $A = \begin{bmatrix} 7 & 5 \\ -3 & -3 \end{bmatrix}$, show that $A^2 - 4A - 6I = O$. Use it to find A^{-1}. [3 marks]

20. Using the characteristic equation, find the inverse of $A = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 3 \\ 2 & 1 & 2 \end{bmatrix}$. [5 marks]

Key sheet

Q	Answer
1	-1
2	$\operatorname{Adj} A = \begin{bmatrix} -8 & -8 & 4 \\ 6 & -2 & -5 \\ 4 & 4 & 2 \end{bmatrix}$
3	$-\frac{1}{12}$
4	$(A^T B)^{-1} = \begin{bmatrix} 4 & 1 \\ 1 & 0 \end{bmatrix}$
5	$x = \pm 6$
6	$\begin{bmatrix} 3 & 5 \\ -1 & -2 \end{bmatrix}$
7	$A^{-1} = \frac{1}{10}\begin{bmatrix} -2 & 4 \\ -1 & -3 \end{bmatrix}$
8	$A^{-1} = \begin{bmatrix} 3 & -2 & 0 \\ 3 & -2 & 1 \\ 2 & -1 & 0 \end{bmatrix}$
9	$\begin{bmatrix} \frac{1}{5} & 0 & 0 \\ 0 & \frac{3}{2} & 0 \\ 0 & 0 & -\frac{1}{2} \end{bmatrix}$
10	$A^{-1} = A^T = \frac{1}{25}\begin{bmatrix} 24 & -7 \\ 7 & 24 \end{bmatrix}$
11	$\operatorname{Adj} A = \begin{bmatrix} -4 & 4 \\ -6 & 3 \end{bmatrix}$; the product is $12I$
12	$\operatorname{Adj} A = \begin{bmatrix} 4 & 7 & -6 \\ 2 & 5 & -6 \\ -2 & 1 & 0 \end{bmatrix}$; both products equal $-6I$
13	64
14	$\lvert A\rvert = 6,\ \lvert\operatorname{Adj} A\rvert = 36 = \lvert A\rvert^2$
15	$125\ \operatorname{Adj} A$
16	$\begin{bmatrix} -3 & -3 & 3 \\ 6 & 3 & 6 \\ -6 & -3 & 3 \end{bmatrix}$
17	$\lambda^3 - 6\lambda^2 + 6\lambda - 4 = 0$
18	$A^{-1} = -\frac{1}{3}(5I - A)$
19	$A^{-1} = \frac{1}{6}\begin{bmatrix} 3 & 5 \\ -3 & -7 \end{bmatrix}$
20	$A^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -8 & -2 & 3 \\ 3 & 1 & -1 \end{bmatrix}$

Solutions

1. If $A = \begin{bmatrix} -4 & 2 & 0 \\ 0 & 0 & -1 \\ 1 & -2 & 4 \end{bmatrix}$, find the cofactor A_{12}. [1]

$\text{Minor} = \begin{vmatrix} 0 & -1 \\ 1 & 4 \end{vmatrix} = 1$

$A_{12} = (-1)^{1+2}(1) = -1$

Ans: -1

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2. If $A = \begin{bmatrix} 1 & 2 & 3 \\ -2 & -2 & -1 \\ 2 & 0 & 4 \end{bmatrix}$, find $\text{Adj } A$. [3]

The cofactors are

$A_{11} = -8,\ A_{12} = 6,\ A_{13} = 4$

$A_{21} = -8,\ A_{22} = -2,\ A_{23} = 4$

$A_{31} = 4,\ A_{32} = -5,\ A_{33} = 2$

The cofactors of each row form a column of the adjoint:

$\text{Adj } A = \begin{bmatrix} -8 & -8 & 4 \\ 6 & -2 & -5 \\ 4 & 4 & 2 \end{bmatrix}$

Ans: $\text{Adj } A = \begin{bmatrix} -8 & -8 & 4 \\ 6 & -2 & -5 \\ 4 & 4 & 2 \end{bmatrix}$

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3. A is a 3×3 matrix with $|A| = -12$. Find $|A^{-1}|$. [1]

$|A^{-1}| = \frac{1}{|A|}$

$= -\frac{1}{12}$

Ans: $-\frac{1}{12}$

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4. If $A^{-1} = \begin{bmatrix} 3 & 5 \\ 1 & 2 \end{bmatrix}$ and $B^{-1} = \begin{bmatrix} 3 & -1 \\ 2 & -1 \end{bmatrix}$, find $(A^T B)^{-1}$. [2]

$(A^T B)^{-1} = B^{-1}(A^T)^{-1} = B^{-1}(A^{-1})^T$

$= \begin{bmatrix} 3 & -1 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 3 & 1 \\ 5 & 2 \end{bmatrix}$

$= \begin{bmatrix} 4 & 1 \\ 1 & 0 \end{bmatrix}$

Ans: $(A^T B)^{-1} = \begin{bmatrix} 4 & 1 \\ 1 & 0 \end{bmatrix}$

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5. Find the values of x for which $\begin{bmatrix} x & 18 \\ 2 & x \end{bmatrix}$ is singular. [2]

$|A| = x^2 - 36 = 0$

$x = \pm 6$

Ans: $x = \pm 6$

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6. Find A^{-1} if $A = \begin{bmatrix} 2 & 5 \\ -1 & -3 \end{bmatrix}$. [1]

$|A| = -1$

Swap the diagonal entries, change the signs of the other two, divide by -1.

$A^{-1} = \begin{bmatrix} 3 & 5 \\ -1 & -2 \end{bmatrix}$

Ans: $\begin{bmatrix} 3 & 5 \\ -1 & -2 \end{bmatrix}$

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7. If $A = \begin{bmatrix} -3 & -4 \\ 1 & -2 \end{bmatrix}$, find A^{-1}. [2]

$|A| = (-3)(-2) - (-4)(1) = 10 \neq 0$

$\text{Adj}\, A = \begin{bmatrix} -2 & 4 \\ -1 & -3 \end{bmatrix}$

$A^{-1} = \frac{1}{10}\begin{bmatrix} -2 & 4 \\ -1 & -3 \end{bmatrix}$

Ans: $A^{-1} = \frac{1}{10}\begin{bmatrix} -2 & 4 \\ -1 & -3 \end{bmatrix}$

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8. Find the inverse of $A = \begin{bmatrix} -1 & 0 & 2 \\ -2 & 0 & 3 \\ -1 & 1 & 0 \end{bmatrix}$. [5]

Cofactors:

$A_{11} = -3,\ A_{12} = -3,\ A_{13} = -2$

$A_{21} = 2,\ A_{22} = 2,\ A_{23} = 1$

$A_{31} = 0,\ A_{32} = -1,\ A_{33} = 0$

$|A| = -1(-3) + 0(-3) + 2(-2) = -1 \neq 0$

$\text{Adj}\, A = \begin{bmatrix} -3 & 2 & 0 \\ -3 & 2 & -1 \\ -2 & 1 & 0 \end{bmatrix}$

$A^{-1} = \frac{\text{Adj}\, A}{|A|} = \begin{bmatrix} 3 & -2 & 0 \\ 3 & -2 & 1 \\ 2 & -1 & 0 \end{bmatrix}$

Ans: $A^{-1} = \begin{bmatrix} 3 & -2 & 0 \\ 3 & -2 & 1 \\ 2 & -1 & 0 \end{bmatrix}$

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9. Write the inverse of the diagonal matrix $\begin{bmatrix} 5 & 0 & 0 \\ 0 & \frac{2}{3} & 0 \\ 0 & 0 & -2 \end{bmatrix}$. [1]

$\text{diag}(a, b, c)^{-1} = \text{diag}\left(\frac{1}{a}, \frac{1}{b}, \frac{1}{c}\right)$

$= \begin{bmatrix} \frac{1}{5} & 0 & 0 \\ 0 & \frac{3}{2} & 0 \\ 0 & 0 & -\frac{1}{2} \end{bmatrix}$

Ans: $\begin{bmatrix} \frac{1}{5} & 0 & 0 \\ 0 & \frac{3}{2} & 0 \\ 0 & 0 & -\frac{1}{2} \end{bmatrix}$

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10. Show that $A = \frac{1}{25}\begin{bmatrix} 24 & 7 \\ -7 & 24 \end{bmatrix}$ is orthogonal and hence find A^{-1}. [3]

$A^T = \frac{1}{25}\begin{bmatrix} 24 & -7 \\ 7 & 24 \end{bmatrix}$

$AA^T = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$

So A is orthogonal, and for an orthogonal matrix $A^{-1} = A^T$.

$A^{-1} = \frac{1}{25}\begin{bmatrix} 24 & -7 \\ 7 & 24 \end{bmatrix}$

Ans: $A^{-1} = A^T = \frac{1}{25}\begin{bmatrix} 24 & -7 \\ 7 & 24 \end{bmatrix}$

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11. If $A = \begin{bmatrix} 3 & -4 \\ 6 & -4 \end{bmatrix}$, find $\operatorname{Adj} A$ and show that $(\operatorname{Adj} A)A = |A|\, I$. [2]

$\operatorname{Adj} A = \begin{bmatrix} -4 & 4 \\ -6 & 3 \end{bmatrix}$

$|A| = 12$

$A(\operatorname{Adj} A) = \begin{bmatrix} 12 & 0 \\ 0 & 12 \end{bmatrix} = 12I$

Ans: $\operatorname{Adj} A = \begin{bmatrix} -4 & 4 \\ -6 & 3 \end{bmatrix}$; **the product is** $12I$

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12. Find the adjoint of $A = \begin{bmatrix} -1 & 1 & 2 \\ -2 & 2 & -2 \\ -2 & 3 & -1 \end{bmatrix}$ and show that $A(\operatorname{Adj} A) = |A|\, I$. [5]

Cofactors:

$A_{11} = 4,\ A_{12} = 2,\ A_{13} = -2$

$A_{21} = 7,\ A_{22} = 5,\ A_{23} = 1$

$A_{31} = -6,\ A_{32} = -6,\ A_{33} = 0$

$\operatorname{Adj} A = \begin{bmatrix} 4 & 7 & -6 \\ 2 & 5 & -6 \\ -2 & 1 & 0 \end{bmatrix}$

$|A| = -1(4) + 1(2) + 2(-2) = -6$

$A(\operatorname{Adj} A) = \begin{bmatrix} -6 & 0 & 0 \\ 0 & -6 & 0 \\ 0 & 0 & -6 \end{bmatrix} = -6I$

$(\operatorname{Adj} A)A = \begin{bmatrix} -6 & 0 & 0 \\ 0 & -6 & 0 \\ 0 & 0 & -6 \end{bmatrix} = -6I$

Ans: $\operatorname{Adj} A = \begin{bmatrix} 4 & 7 & -6 \\ 2 & 5 & -6 \\ -2 & 1 & 0 \end{bmatrix}$; **both products equal** $-6I$

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13. A is a 4×4 matrix with $|A| = 4$. Find $|\operatorname{Adj} A|$. [1]

$|\operatorname{Adj} A| = |A|^{n-1}$

$= (4)^3 = 64$

Ans: 64

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14. If $A = \begin{bmatrix} -2 & 1 & 1 \\ 2 & 2 & 2 \\ 3 & 0 & -1 \end{bmatrix}$, find $|A|$ and $\operatorname{Adj} A$, and show that $|\operatorname{Adj} A| = |A|^2$. [5]

Cofactors:

$A_{11} = -2,\ A_{12} = 8,\ A_{13} = -6$

$A_{21} = 1,\ A_{22} = -1,\ A_{23} = 3$

$A_{31} = 0,\ A_{32} = 6,\ A_{33} = -6$

$\operatorname{Adj} A = \begin{bmatrix} -2 & 1 & 0 \\ 8 & -1 & 6 \\ -6 & 3 & -6 \end{bmatrix}$

$|A| = -2(-2) + 1(8) + 1(-6) = 6$

$|\operatorname{Adj} A| = \begin{vmatrix} -2 & 1 & 0 \\ 8 & -1 & 6 \\ -6 & 3 & -6 \end{vmatrix} = 36$

$|A|^2 = (6)^2 = 36$

So $|\operatorname{Adj} A| = |A|^2$.

Ans: $|A| = 6,\ |\operatorname{Adj} A| = 36 = |A|^2$

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15. If A is a square matrix of order 4, express $\operatorname{Adj}(5A)$ in terms of $\operatorname{Adj} A$. [1]

$\operatorname{Adj}(kA) = k^{n-1}\ \operatorname{Adj} A$

$\operatorname{Adj}(5A) = (5)^3\ \operatorname{Adj} A = 125\ \operatorname{Adj} A$

Ans: $125\ \operatorname{Adj} A$

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16. If $A = \begin{bmatrix} -1 & -1 & 1 \\ 2 & 1 & 2 \\ -2 & -1 & 1 \end{bmatrix}$, find $\operatorname{Adj}(\operatorname{Adj} A)$ without finding $\operatorname{Adj} A$. [2]

$\operatorname{Adj}(\operatorname{Adj} A) = |A|^{n-2} A = |A|\ A$ for $n = 3$

$|A| = -1(3) + (-1)(-6) + 1(0) = 3$

$\operatorname{Adj}(\operatorname{Adj} A) = \begin{bmatrix} -3 & -3 & 3 \\ 6 & 3 & 6 \\ -6 & -3 & 3 \end{bmatrix}$

Ans: $\begin{bmatrix} -3 & -3 & 3 \\ 6 & 3 & 6 \\ -6 & -3 & 3 \end{bmatrix}$

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17. Find the characteristic equation of $A = \begin{bmatrix} 4 & 2 & 2 \\ 0 & 0 & -1 \\ 2 & 2 & 2 \end{bmatrix}$. [3]

$S_1 = 4 + 0 + 2 = 6$

$S_2 = \begin{vmatrix} 4 & 2 \\ 0 & 0 \end{vmatrix} + \begin{vmatrix} 0 & -1 \\ 2 & 2 \end{vmatrix} + \begin{vmatrix} 4 & 2 \\ 2 & 2 \end{vmatrix} = 6$

$S_3 = |A| = 4$

$\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$

$\lambda^3 - 6\lambda^2 + 6\lambda - 4 = 0$

Ans: $\lambda^3 - 6\lambda^2 + 6\lambda - 4 = 0$

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18. If a matrix A satisfies $A^2 - 5A - 3I = O$, express A^{-1} in terms of A and I. [2]

Multiply by A^{-1}: $A - 5I - 3A^{-1} = O$

$A^{-1} = -\frac{1}{3}(5I - A)$

Ans: $A^{-1} = -\frac{1}{3}(5I - A)$

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19. If $A = \begin{bmatrix} 7 & 5 \\ -3 & -3 \end{bmatrix}$, show that $A^2 - 4A - 6I = O$. Use it to find A^{-1}. [3]

$A^2 = \begin{bmatrix} 34 & 20 \\ -12 & -6 \end{bmatrix}$

$A^2 - 4A - 6I = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

Multiply by A^{-1}: $A - 4I - 6A^{-1} = O$

$A^{-1} = -\frac{1}{6}(4I - A) = \frac{1}{6}\begin{bmatrix} 3 & 5 \\ -3 & -7 \end{bmatrix}$

Ans: $A^{-1} = \frac{1}{6}\begin{bmatrix} 3 & 5 \\ -3 & -7 \end{bmatrix}$

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20. Using the characteristic equation, find the inverse of $A = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 3 \\ 2 & 1 & 2 \end{bmatrix}$. [5]

$S_1 = 4,\ S_2 = \begin{vmatrix} 1 & 0 \\ -1 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 3 \\ 1 & 2 \end{vmatrix} + \begin{vmatrix} 1 & 0 \\ 2 & 2 \end{vmatrix} = 2,\ S_3 = |A| = -1$

$\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0:\ \lambda^3 - 4\lambda^2 + 2\lambda + 1 = 0$

$A^3 - 4A^2 + 2A + I = O$

$A^{-1} = \frac{1}{S_3}(A^2 - S_1 A + S_2 I) = -(A^2 - 4A + 2I)$

$A^2 = \begin{bmatrix} 1 & 0 & 0 \\ 4 & 4 & 9 \\ 5 & 3 & 7 \end{bmatrix}$

$A^2 - 4A + 2I = \begin{bmatrix} -1 & 0 & 0 \\ 8 & 2 & -3 \\ -3 & -1 & 1 \end{bmatrix}$

$A^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -8 & -2 & 3 \\ 3 & 1 & -1 \end{bmatrix}$

Ans: $A^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -8 & -2 & 3 \\ 3 & 1 & -1 \end{bmatrix}$