EXPRESSWALA ACADEMY

Maths 1A: Hyperbolic Functions

Question Paper, Set 09

Time: 1 hour 30 minutes | Maximum marks: 50

Name: _____ Class: _____ Date: _____

1. Answer all 20 questions. The marks for each question are shown in brackets.
2. Show your working for the 2, 3 and 5 mark questions.
3. The key sheet and worked solutions are at the end of this paper.

1. Find $\tanh(-\ln 3)$. [1 mark]

2. Find $\cosh(\ln \frac{5}{2}) + \sinh(\ln \frac{5}{3})$. [2 marks]

3. The function $f(x) = \cosh^3 x + x \sinh 5x$ is [1 mark]

4. Find the range of $f(x) = 2\operatorname{sech} x + 5$. [2 marks]

5. If $\cosh x = \frac{37}{12}$, find $\operatorname{sech} x$. [1 mark]

6. If $\sinh x = \frac{20}{21}$, find $\cosh x$ and $\tanh x$. [2 marks]

7. If $\sinh x = \frac{3}{4}$ and $\cosh y = \frac{17}{8}$, with $x, y > 0$, find $\sinh(x+y)$. [3 marks]

8. Prove that $\cosh(x+y) = \cosh x \cosh y + \sinh x \sinh y$. Hence find $\cosh(x+y)$ when $\sinh x = \frac{3}{4}$ and $\sinh y = \frac{4}{3}$. [5 marks]

9. If $\sinh x = \frac{4}{3}$ and $\cosh x = \frac{5}{3}$, find $\cosh 2x$. [1 mark]

10. If $\cosh x = \frac{3}{2}$, find $\cosh 3x$. [2 marks]

11. Prove that $\cosh 3x = 4\cosh^3 x - 3\cosh x$. Hence find $\cosh 3x$ when $\cosh x = \frac{3}{2}$. [5 marks]

12. Find $(\cosh x - \sinh x)^3$ when $x = \ln \frac{1}{2}$. [1 mark]

13. Using the series, write the first three non-zero terms of $\sinh 3x$. [3 marks]

14. If $\sinh x = \frac{15}{8}$ and $\cosh y = \frac{29}{21}$, find $\cosh(x+y)\cosh(x-y)$. [2 marks]

15. Prove that $\cosh(x+y) - \cosh(x-y) = 2\sinh x \sinh y$. Hence find $\cosh(x+y) - \cosh(x-y)$ when $\sinh x = \frac{8}{15}$ and $\sinh y = \frac{7}{24}$ $(x, y > 0)$. [5 marks]

16. Find $\tanh^{-1}\left(\frac{1}{6}\right)$. [2 marks]

17. Find $\operatorname{cosech}^{-1}\left(-\frac{12}{5}\right)$. [3 marks]

18. Prove that $\cosh^{-1} x = \log_e\left(x + \sqrt{x^2-1}\right)$, $x \geq 1$. Hence find $\cosh^{-1}\left(\frac{5}{4}\right)$. [5 marks]

19. Find the domain of $f(x) = \tanh^{-1}\left(\frac{x}{5}\right)$. [1 mark]

20. Express $\cosh^{-1}\left(\frac{5}{3}\right)$ in terms of $\sinh^{-1}$ and of sech^{-1}. [3 marks]

Key sheet

Q	Answer
1	$-\frac{4}{5}$
2	$\frac{119}{60}$
3	even
4	$(5, 7]$
5	$\frac{12}{37}$
6	$\cosh x = \frac{29}{21},\ \tanh x = \frac{20}{29}$
7	$\frac{63}{16}$
8	$\frac{37}{12}$
9	$\frac{41}{9}$
10	9
11	9
12	8
13	$\sinh 3x \approx \frac{81x^5}{40} + \frac{9x^3}{2} + 3x$
14	$\frac{153049}{28224}$
15	$\frac{14}{45}$
16	$\frac{\log_e \frac{7}{5}}{2}$
17	$-\log_e \frac{3}{2}$
18	$\log_e 2$
19	$(-5, 5)$
20	$\sinh^{-1}\left(\frac{4}{3}\right) = \operatorname{sech}^{-1}\left(\frac{3}{5}\right)$

Solutions

1. Find $\tanh(-\ln 3)$. [1]

$\tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}}$

$\tanh(-\ln 3) = -\frac{3 - \frac{1}{3}}{3 + \frac{1}{3}} = -\frac{4}{5}$

Ans: $-\frac{4}{5}$

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2. Find $\cosh(\ln \frac{5}{2}) + \sinh(\ln \frac{5}{3})$. [2]

$\cosh(\ln \frac{5}{2}) = \frac{\frac{5}{2} + \frac{2}{5}}{2} = \frac{29}{20}$

$\sinh(\ln \frac{5}{3}) = \frac{\frac{5}{3} - \frac{3}{5}}{2} = \frac{8}{15}$

$\text{Sum} = \frac{119}{60}$

Ans: $\frac{119}{60}$

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3. The function $f(x) = \cosh^3 x + x \sinh 5x$ is [1]

$\cosh$ and sech are even; $\sinh$ and $\tanh$ are odd.

Replacing x by $-x$: $f(-x) = f(x)$, so f is even.

Ans: even

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4. Find the range of $f(x) = 2\operatorname{sech} x + 5$. [2]

$0 < \operatorname{sech} x \leq 1$.

$5 < f(x) \leq 7$

Range $= (5, 7]$

Ans: $(5, 7]$

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5. If $\cosh x = \frac{37}{12}$, find $\operatorname{sech} x$. [1]

$\operatorname{sech} x = \frac{1}{\cosh x} = \frac{12}{37}$

Ans: $\frac{12}{37}$

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6. If $\sinh x = \frac{20}{21}$, find $\cosh x$ and $\tanh x$. [2]

$\cosh^2 x = 1 + \sinh^2 x = 1 + \frac{400}{441} = \frac{841}{441}$

Since $\cosh x \geq 1$, $\cosh x = \frac{29}{21}$

$\tanh x = \frac{\sinh x}{\cosh x} = \frac{\frac{20}{21}}{\frac{29}{21}} = \frac{20}{29}$

Ans: $\cosh x = \frac{29}{21},\ \tanh x = \frac{20}{29}$

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7. If $\sinh x = \frac{3}{4}$ and $\cosh y = \frac{17}{8}$, with $x, y > 0$, find $\sinh(x + y)$. [3]

$\cosh x = \sqrt{1 + \frac{9}{16}} = \frac{5}{4},\ \sinh y = \sqrt{\frac{289}{64} - 1} = \frac{15}{8}$

$\sinh(x + y) = \sinh x \cosh y + \cosh x \sinh y$

$= \frac{3}{4} \cdot \frac{17}{8} + \frac{5}{4} \cdot \frac{15}{8} = \frac{63}{16}$

Ans: $\frac{63}{16}$

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8. Prove that $\cosh(x+y) = \cosh x \cosh y + \sinh x \sinh y$. Hence find $\cosh(x+y)$ when $\sinh x = \frac{3}{4}$ and $\sinh y = \frac{4}{3}$. [5]

$\text{RHS} = \frac{(e^x+e^{-x})(e^y+e^{-y})+(e^x-e^{-x})(e^y-e^{-y})}{4}$

$= \frac{(e^{x+y}+e^{x-y}+e^{-x+y}+e^{-x-y})+(e^{x+y}-e^{x-y}-e^{-x+y}+e^{-x-y})}{4}$

$= \frac{2e^{x+y}+2e^{-(x+y)}}{4} = \cosh(x+y) = \text{LHS}$

Hence:

$\cosh x = \frac{5}{4},\ \cosh y = \frac{5}{3}$

$\cosh(x+y) = \frac{5}{4}\cdot\frac{5}{3} + \frac{3}{4}\cdot\frac{4}{3} = \frac{37}{12}$

Ans: $\frac{37}{12}$

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9. If $\sinh x = \frac{4}{3}$ and $\cosh x = \frac{5}{3}$, find $\cosh 2x$. [1]

$\cosh 2x = \cosh^2 x + \sinh^2 x = \frac{25}{9} + \frac{16}{9} = \frac{41}{9}$

Ans: $\frac{41}{9}$

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10. If $\cosh x = \frac{3}{2}$, find $\cosh 3x$. [2]

$\cosh 3x = 4\cosh^3 x - 3\cosh x$

$= 4(\frac{3}{2})^3 - 3(\frac{3}{2}) = 9$

Ans: 9

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11. Prove that $\cosh 3x = 4\cosh^3 x - 3\cosh x$. Hence find $\cosh 3x$ when $\cosh x = \frac{3}{2}$. [5]

$\cosh 3x = \cosh(2x+x) = \cosh 2x \cosh x + \sinh 2x \sinh x$

$= (2\cosh^2 x - 1)\cosh x + 2\sinh^2 x \cosh x$

$= 2\cosh^3 x - \cosh x + 2(\cosh^2 x - 1)\cosh x$

$= 4\cosh^3 x - 3\cosh x$

Hence $\cosh 3x = 4(\frac{3}{2})^3 - 3(\frac{3}{2}) = 9$

Ans: 9

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12. Find $(\cosh x - \sinh x)^3$ when $x = \ln\frac{1}{2}$. [1]

$(\cosh x - \sinh x)^3 = e^{-3x}$

$= e^{-3\ln\frac{1}{2}} = \left(\frac{1}{2}\right)^{-3} = 8$

Ans: 8

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13. Using the series, write the first three non-zero terms of $\sinh 3x$. [3]

$\sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots$

Replace x by $3x$:

$\sinh 3x = 3x + \frac{(3x)^3}{6} + \frac{(3x)^5}{120} + \cdots$

$= 3x + \frac{9x^3}{2} + \frac{81x^5}{40} + \cdots$

Ans: $\sinh 3x \approx \frac{81x^5}{40} + \frac{9x^3}{2} + 3x$

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14. If $\sinh x = \frac{15}{8}$ and $\cosh y = \frac{29}{21}$, find $\cosh(x+y)\cosh(x-y)$. [2]

$\cosh(x+y)\cosh(x-y) = \sinh^2 x + \cosh^2 y$

$= \frac{225}{64} + \frac{841}{441} = \frac{153049}{28224}$

Ans: $\frac{153049}{28224}$

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15. Prove that $\cosh(x+y)-\cosh(x-y)=2\sinh x\sinh y$. Hence find $\cosh(x+y)-\cosh(x-y)$ when $\sinh x=\frac{8}{15}$ and $\sinh y=\frac{7}{24}$ $(x,y>0)$. [5]

Subtract the formula for $\cosh(x-y)$ from that for $\cosh(x+y)$: the $\cosh x\cosh y$ terms cancel.

So the left side equals $2\sinh x\sinh y$.

Hence $2\sinh x\sinh y=2\cdot\frac{8}{15}\cdot\frac{7}{24}=\frac{14}{45}$

Ans: $\frac{14}{45}$

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16. Find $\tanh^{-1}\left(\frac{1}{6}\right)$. [2]

$\tanh^{-1}x=\frac{1}{2}\log_e\frac{1+x}{1-x}$

$=\frac{1}{2}\log_e\frac{\frac{7}{6}}{\frac{5}{6}}=\frac{\log_e\frac{7}{5}}{2}$

Ans: $\frac{\log_e\frac{7}{5}}{2}$

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17. Find $\operatorname{cosech}^{-1}\left(-\frac{12}{5}\right)$. [3]

$\operatorname{cosech}^{-1}x=\log_e\frac{1-\sqrt{1+x^2}}{x}$ (for $x<0$)

$\sqrt{1+\frac{144}{25}}=\frac{13}{5}$

$=-\log_e\frac{3}{2}$

Ans: $-\log_e\frac{3}{2}$

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18. Prove that $\cosh^{-1}x=\log_e\left(x+\sqrt{x^2-1}\right),\ x\geq 1$. Hence find $\cosh^{-1}\left(\frac{5}{4}\right)$. [5]

Let $y=\cosh^{-1}x$ $(y\geq 0)$, so $x=\frac{e^y+e^{-y}}{2}$.

Multiply by $2e^y$: $e^{2y}-2xe^y+1=0$.

$e^y=x\pm\sqrt{x^2-1}$.

$y\geq 0$ gives $e^y\geq 1$, so take $e^y=x+\sqrt{x^2-1}$: $y=\log_e\left(x+\sqrt{x^2-1}\right)$.

Hence $\cosh^{-1}\left(\frac{5}{4}\right)=\log_e 2$

Ans: $\log_e 2$

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19. Find the domain of $f(x)=\tanh^{-1}\left(\frac{x}{5}\right)$. [1]

$\tanh^{-1}u$ needs $-1<u<1$.

$-1<\frac{x}{5}<1\Rightarrow -5<x<5$

Ans: $(-5,5)$

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20. Express $\cosh^{-1}\left(\frac{5}{3}\right)$ in terms of $\sinh^{-1}$ and of sech^{-1}. [3]

$\cosh^{-1}x=\sinh^{-1}\sqrt{x^2-1}=\operatorname{sech}^{-1}\frac{1}{x}$ for $x\geq 1$.

$\sqrt{\frac{25}{9}-1}=\frac{4}{3},\ \frac{1}{\frac{5}{3}}=\frac{3}{5}$

$\cosh^{-1}\left(\frac{5}{3}\right)=\sinh^{-1}\left(\frac{4}{3}\right)=\operatorname{sech}^{-1}\left(\frac{3}{5}\right)$

Ans: $\sinh^{-1}\left(\frac{4}{3}\right)=\operatorname{sech}^{-1}\left(\frac{3}{5}\right)$
