EXPRESSWALA ACADEMY

Maths 1A: Hyperbolic Functions

Question Paper, Set 08

Time: 1 hour 30 minutes | Maximum marks: 50

Name: _____ Class: _____ Date: _____

1. Answer all 20 questions. The marks for each question are shown in brackets.
2. Show your working for the 2, 3 and 5 mark questions.
3. The key sheet and worked solutions are at the end of this paper.

1. Find $\sinh(\ln 3)$. [1 mark]

2. Find $\cosh^2(\ln \frac{4}{3}) - \sinh^2(\ln \frac{4}{3}) + \tanh(\ln \frac{5}{2})$. [2 marks]

3. The function $f(x) = x^2 \cosh 3x + x \sinh 3x$ is [1 mark]

4. Find the range of $f(x) = 7\,\mathrm{sech}\, x - 6$. [2 marks]

5. Simplify $\cosh x + \sinh x$ when $x = \ln \frac{5}{3}$. [1 mark]

6. If $\cosh x = \frac{37}{12}$ and x is negative, find $\sinh x$. [2 marks]

7. If $\tanh x = \frac{3}{4}$ and $\tanh y = \frac{3}{5}$, find $\tanh(x - y)$. [3 marks]

8. Prove that $\tanh(x + y) = \frac{\tanh x + \tanh y}{1 + \tanh x \tanh y}$. Hence find $\tanh(x + y)$ when $\tanh x = \frac{3}{4}$ and $\tanh y = \frac{1}{3}$. [5 marks]

9. If $\sinh x = \frac{3}{4}$, find $\cosh 2x$. [1 mark]

10. If $\tanh x = \frac{1}{5}$, find $\cosh 2x$. [2 marks]

11. Prove that $\cosh 2x = \frac{1 + \tanh^2 x}{1 - \tanh^2 x}$. Hence find $\cosh 2x$ when $\tanh x = \frac{2}{5}$. [5 marks]

12. Find $(\cosh x + \sinh x)^3$ when $x = \ln \frac{1}{2}$. [1 mark]

13. If $\tanh x = \frac{1}{5}$, find $\cosh 6x + \sinh 6x$. [3 marks]

14. If $\sinh x = \frac{5}{12}$ and $\cosh y = \frac{29}{21}$, find $\cosh(x + y)\cosh(x - y)$. [2 marks]

15. Prove that $\cosh(x + y)\cosh(x - y) = \sinh^2 x + \cosh^2 y$. Hence find $\cosh(x + y)\cosh(x - y)$ when $\sinh x = \frac{5}{12}$ and $\cosh y = \frac{17}{8}$. [5 marks]

16. Find $\tanh^{-1}\left(\frac{1}{7}\right)$. [2 marks]

17. Find $\cosh^{-1}\left(\frac{17}{15}\right)$. [3 marks]

18. Prove that $\mathrm{sech}^{-1} x = \log_e \frac{1 + \sqrt{1 - x^2}}{x}$, $0 < x \leq 1$. Hence find $\mathrm{sech}^{-1}\left(\frac{8}{17}\right)$. [5 marks]

19. Find the domain of $f(x) = \cosh^{-1}(x - 3)$. [1 mark]

20. Express $\sinh^{-1}\left(\frac{8}{15}\right)$ in terms of $\tanh^{-1}$ and of $\cosh^{-1}$. [3 marks]

Key sheet

Q	Answer
1	$\frac{4}{3}$
2	$\frac{50}{29}$
3	even
4	$(-6, 1]$
5	$\frac{5}{3}$
6	$-\frac{35}{12}$
7	$\frac{3}{11}$
8	$\frac{13}{15}$
9	$\frac{17}{8}$
10	$\frac{13}{12}$
11	$\frac{29}{21}$
12	$\frac{1}{8}$
13	$\frac{27}{8}$
14	$\frac{14681}{7056}$
15	$\frac{2701}{576}$
16	$\frac{\log_e \frac{4}{3}}{2}$
17	$\log_e \frac{5}{3}$
18	$\log_e 4$
19	$[4, \infty)$
20	$\tanh^{-1}\left(\frac{8}{17}\right) = \cosh^{-1}\left(\frac{17}{15}\right)$

Solutions

1. Find $\sinh(\ln 3)$. [1]

$\sinh x = \frac{e^x - e^{-x}}{2}$

$e^{\ln 3} = 3,\ e^{-\ln 3} = \frac{1}{3}$

$\sinh(\ln 3) = \frac{3 - \frac{1}{3}}{2} = \frac{4}{3}$

Ans: $\frac{4}{3}$

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2. Find $\cosh^2(\ln \frac{4}{3}) - \sinh^2(\ln \frac{4}{3}) + \tanh(\ln \frac{5}{2})$. [2]

$\cosh^2 x - \sinh^2 x = 1$

$\tanh(\ln \frac{5}{2}) = \frac{\frac{5}{2} - \frac{2}{5}}{\frac{5}{2} + \frac{2}{5}} = \frac{21}{29}$

$= 1 + \frac{21}{29} = \frac{50}{29}$

Ans: $\frac{50}{29}$

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3. The function $f(x) = x^2 \cosh 3x + x \sinh 3x$ is [1]

cosh and sech are even; sinh and tanh are odd.

Replacing x by $-x$: $f(-x) = f(x)$, so f is even.

Ans: even

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4. Find the range of $f(x) = 7 \operatorname{sech} x - 6$. [2]

$0 < \operatorname{sech} x \leq 1$.

$-6 < f(x) \leq 1$

Range $= (-6, 1]$

Ans: $(-6, 1]$

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5. Simplify $\cosh x + \sinh x$ when $x = \ln \frac{5}{3}$. [1]

$\cosh x + \sinh x = e^x$

$e^{\ln \frac{5}{3}} = \frac{5}{3}$

Ans: $\frac{5}{3}$

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6. If $\cosh x = \frac{37}{12}$ and x is negative, find $\sinh x$. [2]

$\sinh^2 x = \cosh^2 x - 1 = \frac{1369}{144} - 1 = \frac{1225}{144}$

x is negative, so $\sinh x$ is negative: $\sinh x = -\frac{35}{12}$

Ans: $-\frac{35}{12}$

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7. If $\tanh x = \frac{3}{4}$ and $\tanh y = \frac{3}{5}$, find $\tanh(x - y)$. [3]

$\tanh(x - y) = \frac{\tanh x - \tanh y}{1 - \tanh x \tanh y}$

$= \frac{\frac{3}{4} - \frac{3}{5}}{1 - \frac{9}{20}} = \frac{3}{11}$

Ans: $\frac{3}{11}$

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8. Prove that $\tanh(x+y) = \frac{\tanh x + \tanh y}{1+\tanh x \tanh y}$. Hence find $\tanh(x+y)$ when $\tanh x = \frac{3}{4}$ and $\tanh y = \frac{1}{3}$. [5]

$\tanh(x+y) = \frac{\sinh(x+y)}{\cosh(x+y)} = \frac{\sinh x \cosh y + \cosh x \sinh y}{\cosh x \cosh y + \sinh x \sinh y}$

Divide the numerator and the denominator by $\cosh x \cosh y$:

$= \frac{\tanh x + \tanh y}{1+\tanh x \tanh y}$

Hence:

$\tanh(x+y) = \frac{\frac{3}{4}+\frac{1}{3}}{1+\frac{1}{4}} = \frac{13}{15}$

Ans: $\frac{13}{15}$

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9. If $\sinh x = \frac{3}{4}$, find $\cosh 2x$. [1]

$\cosh 2x = 1 + 2\sinh^2 x = 1 + 2 \cdot \frac{9}{16} = \frac{17}{8}$

Ans: $\frac{17}{8}$

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10. If $\tanh x = \frac{1}{5}$, find $\cosh 2x$. [2]

$\cosh 2x = \frac{1+\tanh^2 x}{1-\tanh^2 x} = \frac{1+\frac{1}{25}}{1-\frac{1}{25}} = \frac{13}{12}$

Ans: $\frac{13}{12}$

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11. Prove that $\cosh 2x = \frac{1+\tanh^2 x}{1-\tanh^2 x}$. Hence find $\cosh 2x$ when $\tanh x = \frac{2}{5}$. [5]

$\cosh 2x = \cosh^2 x + \sinh^2 x = \frac{\cosh^2 x + \sinh^2 x}{\cosh^2 x - \sinh^2 x}$

Divide the numerator and the denominator by $\cosh^2 x$:

$= \frac{1+\tanh^2 x}{1-\tanh^2 x}$

Hence $\cosh 2x = \frac{29}{21}$

Ans: $\frac{29}{21}$

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12. Find $(\cosh x + \sinh x)^3$ when $x = \ln \frac{1}{2}$. [1]

$(\cosh x + \sinh x)^3 = e^{3x}$

$= e^{3 \ln \frac{1}{2}} = \left(\frac{1}{2}\right)^3 = \frac{1}{8}$

Ans: $\frac{1}{8}$

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13. If $\tanh x = \frac{1}{5}$, find $\cosh 6x + \sinh 6x$. [3]

$\cosh 2nx + \sinh 2nx = \left(\frac{1+\tanh x}{1-\tanh x}\right)^n$ with $n = 3$

$\frac{1+\frac{1}{5}}{1-\frac{1}{5}} = \frac{3}{2}$

$\text{Answer} = \left(\frac{3}{2}\right)^3 = \frac{27}{8}$

Ans: $\frac{27}{8}$

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14. If $\sinh x = \frac{5}{12}$ and $\cosh y = \frac{29}{21}$, find $\cosh(x+y)\cosh(x-y)$. [2]

$\cosh(x+y)\cosh(x-y) = \sinh^2 x + \cosh^2 y$

$= \frac{25}{144} + \frac{841}{441} = \frac{14681}{7056}$

Ans: $\frac{14681}{7056}$

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15. Prove that $\cosh(x+y)\cosh(x-y) = \sinh^2 x + \cosh^2 y$. Hence find $\cosh(x+y)\cosh(x-y)$ when $\sinh x = \frac{5}{12}$ and $\cosh y = \frac{17}{8}$. [5]

Multiply: $(\cosh x \cosh y + \sinh x \sinh y)(\cosh x \cosh y - \sinh x \sinh y) = \cosh^2 x \cosh^2 y - \sinh^2 x \sinh^2 y$.

Put $\cosh^2 x = 1 + \sinh^2 x$ and $\sinh^2 y = \cosh^2 y - 1$:

$= (1+\sinh^2 x)\cosh^2 y - \sinh^2 x(\cosh^2 y - 1) = \sinh^2 x + \cosh^2 y$

Hence $\frac{25}{144} + \frac{289}{64} = \frac{2701}{576}$

Ans: $\frac{2701}{576}$

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16. Find $\tanh^{-1}\left(\frac{1}{7}\right)$. [2]

$\tanh^{-1} x = \frac{1}{2}\log_e \frac{1+x}{1-x}$

$= \frac{1}{2}\log_e \frac{\frac{8}{7}}{\frac{6}{7}} = \frac{\log_e \frac{4}{3}}{2}$

Ans: $\frac{\log_e \frac{4}{3}}{2}$

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17. Find $\cosh^{-1}\left(\frac{17}{15}\right)$. [3]

$\cosh^{-1} x = \log_e\left(x + \sqrt{x^2-1}\right)$

$\sqrt{\frac{289}{225} - 1} = \frac{8}{15}$

$\cosh^{-1}\left(\frac{17}{15}\right) = \log_e\left(\frac{17}{15} + \frac{8}{15}\right) = \log_e \frac{5}{3}$

Ans: $\log_e \frac{5}{3}$

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18. Prove that $\operatorname{sech}^{-1} x = \log_e \frac{1+\sqrt{1-x^2}}{x}$, $0 < x \leq 1$. Hence find $\operatorname{sech}^{-1}\left(\frac{8}{17}\right)$. [5]

Let $y = \operatorname{sech}^{-1} x$ ($y \geq 0$), so $\cosh y = \frac{1}{x}$.

$y = \cosh^{-1}\frac{1}{x} = \log_e\left(\frac{1}{x} + \sqrt{\frac{1}{x^2}-1}\right)$.

$= \log_e \frac{1+\sqrt{1-x^2}}{x}$.

Hence $\operatorname{sech}^{-1}\left(\frac{8}{17}\right) = \log_e 4$

Ans: $\log_e 4$

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19. Find the domain of $f(x) = \cosh^{-1}(x-3)$. [1]

$\cosh^{-1} u$ needs $u \geq 1$.

$x - 3 \geq 1 \Rightarrow x \geq 4$

Ans: $[4, \infty)$

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20. Express $\sinh^{-1}\left(\frac{8}{15}\right)$ in terms of $\tanh^{-1}$ and of $\cosh^{-1}$. [3]

$\sinh^{-1} x = \tanh^{-1}\frac{x}{\sqrt{1+x^2}} = \cosh^{-1}\sqrt{x^2+1}$ for $x > 0$.

$\sqrt{1 + \frac{64}{225}} = \frac{17}{15}$

$\tanh^{-1}\frac{\frac{8}{15}}{\frac{17}{15}} = \tanh^{-1}\left(\frac{8}{17}\right)$

$\sinh^{-1}\left(\frac{8}{15}\right) = \tanh^{-1}\left(\frac{8}{17}\right) = \cosh^{-1}\left(\frac{17}{15}\right)$

Ans: $\tanh^{-1}\left(\frac{8}{17}\right) = \cosh^{-1}\left(\frac{17}{15}\right)$
