EXPRESSWALA ACADEMY

Maths 1A: Hyperbolic Functions

Question Paper, Set 07

Time: 1 hour 30 minutes | Maximum marks: 50

Name: _____ Class: _____ Date: _____

1. Answer all 20 questions. The marks for each question are shown in brackets.
2. Show your working for the 2, 3 and 5 mark questions.
3. The key sheet and worked solutions are at the end of this paper.

1. Find $\cosh(\ln \frac{4}{3})$. [1 mark]

2. Find $\coth(\ln \frac{4}{3})$. [2 marks]

3. The function $f(x) = \cosh^5 x + x \sinh 2x$ is [1 mark]

4. Find the range of $f(x) = 2\tanh x + 1$. [2 marks]

5. Simplify $\cosh x + \sinh x$ when $x = \ln 3$. [1 mark]

6. If $\tanh x = \frac{3}{5}$, find $\operatorname{sech} x$. [2 marks]

7. If $\tanh x = \frac{1}{5}$ and $\tanh y = \frac{2}{7}$, find $\tanh(x + y)$. [3 marks]

8. Prove that $\sinh(x - y) = \sinh x \cosh y - \cosh x \sinh y$. Hence find $\sinh(x - y)$ when $\sinh x = \frac{5}{12}$ and $\cosh y = \frac{5}{4}$ $(x, y > 0)$. [5 marks]

9. If $\sinh x = \frac{35}{12}$ and $\cosh x = \frac{37}{12}$, find $\sinh 2x$. [1 mark]

10. If $\tanh x = \frac{2}{5}$, find $\tanh 2x$. [2 marks]

11. Prove that $\sinh 2x = 2\sinh x \cosh x$. Hence find $\sinh 2x$ when $\sinh x = \frac{12}{5}$ $(x > 0)$. [5 marks]

12. Find $(\cosh x - \sinh x)^2$ when $x = \ln \frac{3}{2}$. [1 mark]

13. If $\tanh x = \frac{1}{4}$, find $\cosh 2x + \sinh 2x$. [3 marks]

14. If $\sinh x = \frac{20}{21}$ and $\sinh y = \frac{5}{12}$, find $\sinh(x + y)\sinh(x - y)$. [2 marks]

15. Prove that $\cosh(x + y) - \cosh(x - y) = 2\sinh x \sinh y$. Hence find $\cosh(x + y) - \cosh(x - y)$ when $\sinh x = \frac{20}{21}$ and $\sinh y = \frac{4}{3}$ $(x, y > 0)$. [5 marks]

16. Find $\tanh^{-1}\left(\frac{3}{7}\right)$. [2 marks]

17. Find $\cosh^{-1}\left(\frac{29}{21}\right)$. [3 marks]

18. Prove that $\coth^{-1} x = \frac{1}{2}\log_e \frac{x+1}{x-1}$, $|x| > 1$. Hence find $\coth^{-1}(4)$. [5 marks]

19. Find the domain of $f(x) = \cosh^{-1}(x - 9)$. [1 mark]

20. Express $\sinh^{-1}\left(\frac{3}{4}\right)$ in terms of $\tanh^{-1}$ and of $\cosh^{-1}$. [3 marks]

Key sheet

Q	Answer
1	$\frac{25}{24}$
2	$\frac{25}{7}$
3	even
4	$(-1, 3)$
5	3
6	$\frac{4}{5}$
7	$\frac{17}{37}$
8	$-\frac{7}{24}$
9	$\frac{1295}{72}$
10	$\frac{20}{29}$
11	$\frac{312}{25}$
12	$\frac{4}{9}$
13	$\frac{5}{3}$
14	$\frac{575}{784}$
15	$\frac{160}{63}$
16	$\frac{\log_e \frac{5}{2}}{2}$
17	$\log_e \frac{7}{3}$
18	$\frac{\log_e \frac{5}{3}}{2}$
19	$[10, \infty)$
20	$\tanh^{-1}\left(\frac{3}{5}\right) = \cosh^{-1}\left(\frac{5}{4}\right)$

Solutions

1. Find $\cosh(\ln \frac{4}{3})$. [1]

$\cosh x = \frac{e^x + e^{-x}}{2}$

$\cosh(\ln \frac{4}{3}) = \frac{\frac{4}{3} + \frac{3}{4}}{2} = \frac{25}{24}$

Ans: $\frac{25}{24}$

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2. Find $\coth(\ln \frac{4}{3})$. [2]

$\coth x = \frac{e^x + e^{-x}}{e^x - e^{-x}}$

$e^{\ln \frac{4}{3}} = \frac{4}{3},\ e^{-\ln \frac{4}{3}} = \frac{3}{4}$

$\coth(\ln \frac{4}{3}) = \frac{25}{7}$

Ans: $\frac{25}{7}$

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3. The function $f(x) = \cosh^5 x + x \sinh 2x$ is [1]

$\cosh$ and sech are even; $\sinh$ and $\tanh$ are odd.

Replacing x by $-x$: $f(-x) = f(x)$, so f is even.

Ans: even

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4. Find the range of $f(x) = 2\tanh x + 1$. [2]

$-1 < \tanh x < 1$.

$1 - 2 < f(x) < 1 + 2$

Range $= (-1, 3)$

Ans: $(-1, 3)$

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5. Simplify $\cosh x + \sinh x$ when $x = \ln 3$. [1]

$\cosh x + \sinh x = e^x$

$e^{\ln 3} = 3$

Ans: 3

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6. If $\tanh x = \frac{3}{5}$, find $\operatorname{sech} x$. [2]

$\operatorname{sech}^2 x = 1 - \tanh^2 x = 1 - \frac{9}{25} = \frac{16}{25}$

$\operatorname{sech} x > 0$, so $\operatorname{sech} x = \frac{4}{5}$

Ans: $\frac{4}{5}$

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7. If $\tanh x = \frac{1}{5}$ and $\tanh y = \frac{2}{7}$, find $\tanh(x + y)$. [3]

$\tanh(x + y) = \frac{\tanh x + \tanh y}{1 + \tanh x \tanh y}$

$= \frac{\frac{1}{5} + \frac{2}{7}}{1 + \frac{2}{35}} = \frac{17}{37}$

Ans: $\frac{17}{37}$

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8. Prove that $\sinh(x-y) = \sinh x \cosh y - \cosh x \sinh y$. Hence find $\sinh(x-y)$ when $\sinh x = \frac{5}{12}$ and $\cosh y = \frac{5}{4}$ $(x, y > 0)$. [5]

$\text{RHS} = \frac{(e^x - e^{-x})(e^y + e^{-y}) - (e^x + e^{-x})(e^y - e^{-y})}{4}$

$= \frac{(e^{x+y} + e^{x-y} - e^{-x+y} - e^{-x-y}) - (e^{x+y} - e^{x-y} + e^{-x+y} - e^{-x-y})}{4}$

$= \frac{2e^{x-y} - 2e^{-(x-y)}}{4} = \sinh(x-y) = \text{LHS}$

Hence:

$\cosh x = \frac{13}{12},\ \sinh y = \frac{3}{4}$

$\sinh(x-y) = \frac{5}{12} \cdot \frac{5}{4} - \frac{13}{12} \cdot \frac{3}{4} = -\frac{7}{24}$

Ans: $-\frac{7}{24}$

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9. If $\sinh x = \frac{35}{12}$ and $\cosh x = \frac{37}{12}$, find $\sinh 2x$. [1]

$\sinh 2x = 2 \sinh x \cosh x = 2 \cdot \frac{35}{12} \cdot \frac{37}{12} = \frac{1295}{72}$

Ans: $\frac{1295}{72}$

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10. If $\tanh x = \frac{2}{5}$, find $\tanh 2x$. [2]

$\tanh 2x = \frac{2 \tanh x}{1 + \tanh^2 x} = \frac{\frac{4}{5}}{1 + \frac{4}{25}} = \frac{20}{29}$

Ans: $\frac{20}{29}$

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11. Prove that $\sinh 2x = 2 \sinh x \cosh x$. Hence find $\sinh 2x$ when $\sinh x = \frac{12}{5}$ $(x > 0)$. [5]

$2 \sinh x \cosh x = 2 \cdot \frac{e^x - e^{-x}}{2} \cdot \frac{e^x + e^{-x}}{2}$

$= \frac{e^{2x} - e^{-2x}}{2} = \sinh 2x$

Hence $\cosh x = \frac{13}{5},\ \sinh 2x = 2 \cdot \frac{12}{5} \cdot \frac{13}{5} = \frac{312}{25}$

Ans: $\frac{312}{25}$

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12. Find $(\cosh x - \sinh x)^2$ when $x = \ln \frac{3}{2}$. [1]

$(\cosh x - \sinh x)^2 = e^{-2x}$

$= e^{-2 \ln \frac{3}{2}} = \left(\frac{3}{2}\right)^{-2} = \frac{4}{9}$

Ans: $\frac{4}{9}$

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13. If $\tanh x = \frac{1}{4}$, find $\cosh 2x + \sinh 2x$. [3]

$\cosh 2nx + \sinh 2nx = \left(\frac{1 + \tanh x}{1 - \tanh x}\right)^n$ with $n = 1$

$\frac{1 + \frac{1}{4}}{1 - \frac{1}{4}} = \frac{5}{3}$

$\text{Answer} = \left(\frac{5}{3}\right)^1 = \frac{5}{3}$

Ans: $\frac{5}{3}$

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14. If $\sinh x = \frac{20}{21}$ and $\sinh y = \frac{5}{12}$, find $\sinh(x+y)\sinh(x-y)$. [2]

$\sinh(x+y)\sinh(x-y) = \sinh^2 x - \sinh^2 y$

$= \frac{400}{441} - \frac{25}{144} = \frac{575}{784}$

Ans: $\frac{575}{784}$

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15. Prove that $\cosh(x+y)-\cosh(x-y)=2\sinh x\sinh y$. Hence find $\cosh(x+y)-\cosh(x-y)$ when $\sinh x=\frac{20}{21}$ and $\sinh y=\frac{4}{3}$ $(x,y>0)$. [5]

Subtract the formula for $\cosh(x-y)$ from that for $\cosh(x+y)$: the $\cosh x\cosh y$ terms cancel.

So the left side equals $2\sinh x\sinh y$.

Hence $2\sinh x\sinh y=2\cdot\frac{20}{21}\cdot\frac{4}{3}=\frac{160}{63}$

Ans: $\frac{160}{63}$

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16. Find $\tanh^{-1}\left(\frac{3}{7}\right)$. [2]

$\tanh^{-1}x=\frac{1}{2}\log_e\frac{1+x}{1-x}$

$=\frac{1}{2}\log_e\frac{\frac{10}{7}}{\frac{4}{7}}=\frac{\log_e\frac{5}{2}}{2}$

Ans: $\frac{\log_e\frac{5}{2}}{2}$

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17. Find $\cosh^{-1}\left(\frac{29}{21}\right)$. [3]

$\cosh^{-1}x=\log_e\left(x+\sqrt{x^2-1}\right)$

$\sqrt{\frac{841}{441}-1}=\frac{20}{21}$

$\cosh^{-1}\left(\frac{29}{21}\right)=\log_e\left(\frac{29}{21}+\frac{20}{21}\right)=\log_e\frac{7}{3}$

Ans: $\log_e\frac{7}{3}$

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18. Prove that $\coth^{-1}x=\frac{1}{2}\log_e\frac{x+1}{x-1},\ |x|>1$. Hence find $\coth^{-1}(4)$. [5]

Let $y=\coth^{-1}x$, so $x=\frac{e^{2y}+1}{e^{2y}-1}$.

$xe^{2y}-x=e^{2y}+1\Rightarrow e^{2y}(x-1)=x+1.$

$e^{2y}=\frac{x+1}{x-1}\Rightarrow y=\frac{1}{2}\log_e\frac{x+1}{x-1}.$

Hence $\coth^{-1}(4)=\frac{\log_e\frac{5}{3}}{2}$

Ans: $\frac{\log_e\frac{5}{3}}{2}$

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19. Find the domain of $f(x)=\cosh^{-1}(x-9)$. [1]

$\cosh^{-1}u$ needs $u\geq 1$.

$x-9\geq 1\Rightarrow x\geq 10$

Ans: $[10,\infty)$

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20. Express $\sinh^{-1}\left(\frac{3}{4}\right)$ in terms of $\tanh^{-1}$ and of $\cosh^{-1}$. [3]

$\sinh^{-1}x=\tanh^{-1}\frac{x}{\sqrt{1+x^2}}=\cosh^{-1}\sqrt{x^2+1}$ for $x>0$.

$\sqrt{1+\frac{9}{16}}=\frac{5}{4}$

$\tanh^{-1}\frac{\frac{3}{4}}{\frac{5}{4}}=\tanh^{-1}\left(\frac{3}{5}\right)$

$\sinh^{-1}\left(\frac{3}{4}\right)=\tanh^{-1}\left(\frac{3}{5}\right)=\cosh^{-1}\left(\frac{5}{4}\right)$

Ans: $\tanh^{-1}\left(\frac{3}{5}\right)=\cosh^{-1}\left(\frac{5}{4}\right)$
