EXPRESSWALA ACADEMY

# Maths 1A: Hyperbolic Functions

## Question Paper, Set 06

Time: 1 hour 30 minutes | Maximum marks: 50

Name: _____ Class: _____ Date: _____

1. Answer all 20 questions. The marks for each question are shown in brackets.
2. Show your working for the 2, 3 and 5 mark questions.
3. The key sheet and worked solutions are at the end of this paper.

**1.** Find $\cosh(\ln \frac{3}{2})$. [1 mark]

**2.** Find $\cosh(\ln 6) + \sinh(\ln \frac{4}{3})$. [2 marks]

**3.** The function $f(x) = x\cosh 3x + \tanh 3x$ is [1 mark]

**4.** Find the range of $f(x) = 6\,\mathrm{sech}\, x + 4$. [2 marks]

**5.** If $\sinh x = \frac{4}{3}$, find $\mathrm{cosech}\, x$. [1 mark]

**6.** If $\tanh x = -\frac{8}{17}$, find $\mathrm{sech}\, x$. [2 marks]

**7.** If $\tanh x = \frac{1}{6}$ and $\tanh y = \frac{3}{7}$, find $\tanh(x+y)$. [3 marks]

**8.** Prove that $\cosh(x+y) = \cosh x \cosh y + \sinh x \sinh y$. Hence find $\cosh(x+y)$ when $\sinh x = \frac{21}{20}$ and $\sinh y = \frac{8}{15}$. [5 marks]

**9.** If $\cosh x = \frac{17}{15}$, find $\cosh 2x$. [1 mark]

**10.** If $\sinh x = 1$, find $\sinh 3x$. [2 marks]

**11.** Prove that $\sinh 2x = 2\sinh x \cosh x$. Hence find $\sinh 2x$ when $\sinh x = \frac{3}{4}$ $(x > 0)$. [5 marks]

**12.** Find $(\cosh x - \sinh x)^3$ when $x = \ln \frac{1}{3}$. [1 mark]

**13.** If $\tanh x = \frac{3}{7}$, find $\cosh 2x + \sinh 2x$. [3 marks]

**14.** If $\sinh x = \frac{8}{15}$ and $\sinh y = \frac{35}{12}$, find $\sinh(x+y)\sinh(x-y)$. [2 marks]

**15.** Prove that $\cosh(x+y)\cosh(x-y) = \sinh^2 x + \cosh^2 y$. Hence find $\cosh(x+y)\cosh(x-y)$ when $\sinh x = \frac{35}{12}$ and $\cosh y = \frac{13}{12}$. [5 marks]

**16.** Find $\coth^{-1}(4)$. [2 marks]

**17.** Find $\mathrm{sech}^{-1}\left(\frac{24}{25}\right)$. [3 marks]

**18.** Prove that $\mathrm{sech}^{-1} x = \log_e \frac{1+\sqrt{1-x^2}}{x}$, $0 < x \leq 1$. Hence find $\mathrm{sech}^{-1}\left(\frac{20}{29}\right)$. [5 marks]

**19.** Find the domain of $f(x) = \tanh^{-1}\left(\frac{x}{3}\right)$. [1 mark]

**20.** Express $\sinh^{-1}\left(\frac{20}{21}\right)$ in terms of $\tanh^{-1}$ and of $\cosh^{-1}$. [3 marks]

## Key sheet

| Q | Answer |
|---|---|
| 1 | $\frac{13}{12}$ |
| 2 | $\frac{27}{8}$ |
| 3 | odd |
| 4 | $(4, 10]$ |
| 5 | $\frac{3}{4}$ |
| 6 | $\frac{15}{17}$ |
| 7 | $\frac{5}{9}$ |
| 8 | $\frac{661}{300}$ |
| 9 | $\frac{353}{225}$ |
| 10 | $7$ |
| 11 | $\frac{15}{8}$ |
| 12 | $27$ |
| 13 | $\frac{5}{2}$ |
| 14 | $-\frac{3289}{400}$ |
| 15 | $\frac{697}{72}$ |
| 16 | $\frac{\log_e \frac{5}{3}}{2}$ |
| 17 | $\log_e \frac{4}{3}$ |
| 18 | $\log_e \frac{5}{2}$ |
| 19 | $(-3, 3)$ |
| 20 | $\tanh^{-1}\left(\frac{20}{29}\right) = \cosh^{-1}\left(\frac{29}{21}\right)$ |

## Solutions

**1.** Find $\cosh(\ln \frac{3}{2})$. [1]

$\cosh x = \frac{e^x + e^{-x}}{2}$

$\cosh(\ln \frac{3}{2}) = \frac{\frac{3}{2} + \frac{2}{3}}{2} = \frac{13}{12}$

**Ans:** $\frac{13}{12}$

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**2.** Find $\cosh(\ln 6) + \sinh(\ln \frac{4}{3})$. [2]

$\cosh(\ln 6) = \frac{6 + \frac{1}{6}}{2} = \frac{37}{12}$

$\sinh(\ln \frac{4}{3}) = \frac{\frac{4}{3} - \frac{3}{4}}{2} = \frac{7}{24}$

$\text{Sum} = \frac{27}{8}$

**Ans:** $\frac{27}{8}$

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**3.** The function $f(x) = x \cosh 3x + \tanh 3x$ is [1]

$\cosh$ and $\text{sech}$ are even; $\sinh$ and $\tanh$ are odd.

Replacing $x$ by $-x$: $f(-x) = -f(x)$, so $f$ is odd.

**Ans: odd**

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**4.** Find the range of $f(x) = 6\,\text{sech}\, x + 4$. [2]

$0 < \text{sech}\, x \leq 1$.

$4 < f(x) \leq 10$

Range $= (4, 10]$

**Ans:** $(4, 10]$

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**5.** If $\sinh x = \frac{4}{3}$, find $\text{cosech}\, x$. [1]

$\text{cosech}\, x = \frac{1}{\sinh x} = \frac{3}{4}$

**Ans:** $\frac{3}{4}$

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**6.** If $\tanh x = -\frac{8}{17}$, find $\text{sech}\, x$. [2]

$\text{sech}^2 x = 1 - \tanh^2 x = 1 - \frac{64}{289} = \frac{225}{289}$

$\text{sech}\, x > 0$, so $\text{sech}\, x = \frac{15}{17}$

**Ans:** $\frac{15}{17}$

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**7.** If $\tanh x = \frac{1}{6}$ and $\tanh y = \frac{3}{7}$, find $\tanh(x + y)$. [3]

$\tanh(x + y) = \frac{\tanh x + \tanh y}{1 + \tanh x \tanh y}$

$= \frac{\frac{1}{6} + \frac{3}{7}}{1 + \frac{1}{14}} = \frac{5}{9}$

**Ans:** $\frac{5}{9}$

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**8.** Prove that $\cosh(x+y) = \cosh x \cosh y + \sinh x \sinh y$. Hence find $\cosh(x+y)$ when $\sinh x = \frac{21}{20}$ and $\sinh y = \frac{8}{15}$. [5]

$\text{RHS} = \frac{(e^x+e^{-x})(e^y+e^{-y})+(e^x-e^{-x})(e^y-e^{-y})}{4}$

$= \frac{(e^{x+y}+e^{x-y}+e^{-x+y}+e^{-x-y})+(e^{x+y}-e^{x-y}-e^{-x+y}+e^{-x-y})}{4}$

$= \frac{2e^{x+y}+2e^{-(x+y)}}{4} = \cosh(x+y) = \text{LHS}$

Hence:

$\cosh x = \frac{29}{20},\ \cosh y = \frac{17}{15}$

$\cosh(x+y) = \frac{29}{20}\cdot\frac{17}{15} + \frac{21}{20}\cdot\frac{8}{15} = \frac{661}{300}$

**Ans:** $\frac{661}{300}$

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**9.** If $\cosh x = \frac{17}{15}$, find $\cosh 2x$. [1]

$\cosh 2x = 2\cosh^2 x - 1 = 2\cdot\frac{289}{225} - 1 = \frac{353}{225}$

**Ans:** $\frac{353}{225}$

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**10.** If $\sinh x = 1$, find $\sinh 3x$. [2]

$\sinh 3x = 3\sinh x + 4\sinh^3 x$

$= 3(1) + 4(1)^3 = 7$

**Ans:** $7$

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**11.** Prove that $\sinh 2x = 2\sinh x \cosh x$. Hence find $\sinh 2x$ when $\sinh x = \frac{3}{4}$ $(x > 0)$. [5]

$2\sinh x \cosh x = 2\cdot\frac{e^x-e^{-x}}{2}\cdot\frac{e^x+e^{-x}}{2}$

$= \frac{e^{2x}-e^{-2x}}{2} = \sinh 2x$

Hence $\cosh x = \frac{5}{4},\ \sinh 2x = 2\cdot\frac{3}{4}\cdot\frac{5}{4} = \frac{15}{8}$

**Ans:** $\frac{15}{8}$

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**12.** Find $(\cosh x - \sinh x)^3$ when $x = \ln\frac{1}{3}$. [1]

$(\cosh x - \sinh x)^3 = e^{-3x}$

$= e^{-3\ln\frac{1}{3}} = \left(\frac{1}{3}\right)^{-3} = 27$

**Ans:** $27$

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**13.** If $\tanh x = \frac{3}{7}$, find $\cosh 2x + \sinh 2x$. [3]

$\cosh 2nx + \sinh 2nx = \left(\frac{1+\tanh x}{1-\tanh x}\right)^n$ with $n = 1$

$\frac{1+\frac{3}{7}}{1-\frac{3}{7}} = \frac{5}{2}$

$\text{Answer} = \left(\frac{5}{2}\right)^1 = \frac{5}{2}$

**Ans:** $\frac{5}{2}$

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**14.** If $\sinh x = \frac{8}{15}$ and $\sinh y = \frac{35}{12}$, find $\sinh(x+y)\sinh(x-y)$. [2]

$\sinh(x+y)\sinh(x-y) = \sinh^2 x - \sinh^2 y$

$= \frac{64}{225} - \frac{1225}{144} = -\frac{3289}{400}$

**Ans:** $-\frac{3289}{400}$

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**15.** Prove that $\cosh(x+y)\cosh(x-y) = \sinh^2 x + \cosh^2 y$. Hence find $\cosh(x+y)\cosh(x-y)$ when $\sinh x = \frac{35}{12}$ and $\cosh y = \frac{13}{12}$. [5]

Multiply: $(\cosh x \cosh y + \sinh x \sinh y)(\cosh x \cosh y - \sinh x \sinh y) = \cosh^2 x \cosh^2 y - \sinh^2 x \sinh^2 y$.

Put $\cosh^2 x = 1 + \sinh^2 x$ and $\sinh^2 y = \cosh^2 y - 1$:

$= (1 + \sinh^2 x)\cosh^2 y - \sinh^2 x(\cosh^2 y - 1) = \sinh^2 x + \cosh^2 y$

Hence $\frac{1225}{144} + \frac{169}{144} = \frac{697}{72}$

**Ans:** $\frac{697}{72}$

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**16.** Find $\coth^{-1}(4)$. [2]

$\coth^{-1} x = \frac{1}{2}\log_e \frac{x+1}{x-1}$

$= \frac{1}{2}\log_e \frac{5}{3} = \frac{\log_e \frac{5}{3}}{2}$

**Ans:** $\frac{\log_e \frac{5}{3}}{2}$

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**17.** Find $\operatorname{sech}^{-1}\left(\frac{24}{25}\right)$. [3]

$\operatorname{sech}^{-1} x = \log_e \frac{1+\sqrt{1-x^2}}{x}$

$\sqrt{1 - \frac{576}{625}} = \frac{7}{25}$

$= \log_e \frac{1+\frac{7}{25}}{\frac{24}{25}} = \log_e \frac{4}{3}$

**Ans:** $\log_e \frac{4}{3}$

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**18.** Prove that $\operatorname{sech}^{-1} x = \log_e \frac{1+\sqrt{1-x^2}}{x}$, $0 < x \leq 1$. Hence find $\operatorname{sech}^{-1}\left(\frac{20}{29}\right)$. [5]

Let $y = \operatorname{sech}^{-1} x$ ($y \geq 0$), so $\cosh y = \frac{1}{x}$.

$y = \cosh^{-1}\frac{1}{x} = \log_e\left(\frac{1}{x} + \sqrt{\frac{1}{x^2} - 1}\right)$.

$= \log_e \frac{1+\sqrt{1-x^2}}{x}$.

Hence $\operatorname{sech}^{-1}\left(\frac{20}{29}\right) = \log_e \frac{5}{2}$

**Ans:** $\log_e \frac{5}{2}$

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**19.** Find the domain of $f(x) = \tanh^{-1}\left(\frac{x}{3}\right)$. [1]

$\tanh^{-1} u$ needs $-1 < u < 1$.

$-1 < \frac{x}{3} < 1 \Rightarrow -3 < x < 3$

**Ans:** $(-3, 3)$

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**20.** Express $\sinh^{-1}\left(\frac{20}{21}\right)$ in terms of $\tanh^{-1}$ and of $\cosh^{-1}$. [3]

$\sinh^{-1} x = \tanh^{-1}\frac{x}{\sqrt{1+x^2}} = \cosh^{-1}\sqrt{x^2+1}$ for $x > 0$.

$\sqrt{1 + \frac{400}{441}} = \frac{29}{21}$

$\tanh^{-1}\frac{\frac{20}{21}}{\frac{29}{21}} = \tanh^{-1}\left(\frac{20}{29}\right)$

$\sinh^{-1}\left(\frac{20}{21}\right) = \tanh^{-1}\left(\frac{20}{29}\right) = \cosh^{-1}\left(\frac{29}{21}\right)$

**Ans:** $\tanh^{-1}\left(\frac{20}{29}\right) = \cosh^{-1}\left(\frac{29}{21}\right)$

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