EXPRESSWALA ACADEMY

Maths 1A: Hyperbolic Functions

Question Paper, Set 05

Time: 1 hour 30 minutes | Maximum marks: 50

Name: _____ Class: _____ Date: _____

1. Answer all 20 questions. The marks for each question are shown in brackets.
2. Show your working for the 2, 3 and 5 mark questions.
3. The key sheet and worked solutions are at the end of this paper.

1. Find $\cosh(\ln \frac{5}{3})$. [1 mark]

2. Find $\cosh^2(\ln \frac{5}{2}) - \sinh^2(\ln \frac{5}{2}) + \tanh(\ln 6)$. [2 marks]

3. The function $f(x) = \cosh^4 x + x^6 \sinh 3x$ is [1 mark]

4. Find the range of $f(x) = 4\operatorname{sech} x - 6$. [2 marks]

5. If $\tanh x = \frac{7}{25}$, find $\coth x$. [1 mark]

6. If $\sinh x = \frac{7}{24}$, find $\cosh x$ and $\tanh x$. [2 marks]

7. If $\sinh x = \frac{5}{12}$ and $\cosh y = \frac{17}{15}$, with $x, y > 0$, find $\sinh(x - y)$. [3 marks]

8. Prove that $\tanh(x - y) = \frac{\tanh x - \tanh y}{1 - \tanh x \tanh y}$. Hence find $\tanh(x - y)$ when $\tanh x = \frac{1}{4}$ and $\tanh y = \frac{3}{4}$. [5 marks]

9. If $\sinh x = \frac{15}{8}$ and $\cosh x = \frac{17}{8}$, find $\sinh 2x$. [1 mark]

10. If $\tanh x = \frac{1}{6}$, find $\sinh 2x$. [2 marks]

11. Prove that $\sinh 2x = \frac{2\tanh x}{1 - \tanh^2 x}$. Hence find $\sinh 2x$ when $\tanh x = \frac{1}{5}$. [5 marks]

12. Find $(\cosh x + \sinh x)^5$ when $x = \ln \frac{1}{3}$. [1 mark]

13. If $\tanh x = \frac{1}{5}$, find $\cosh 2x + \sinh 2x$. [3 marks]

14. If $\sinh x = \frac{21}{20}$ and $\cosh y = \frac{13}{12}$, find $\cosh(x + y)\cosh(x - y)$. [2 marks]

15. Prove that $\sinh(x + y) - \sinh(x - y) = 2\cosh x \sinh y$. Hence find $\sinh(x + y) - \sinh(x - y)$ when $\cosh x = \frac{37}{12}$ and $\sinh y = \frac{5}{12}$ $(x, y > 0)$. [5 marks]

16. Find $\coth^{-1}(6)$. [2 marks]

17. Find $\operatorname{cosech}^{-1}\left(\frac{5}{12}\right)$. [3 marks]

18. Prove that $\tanh^{-1} x = \frac{1}{2}\log_e \frac{1+x}{1-x}$, $|x| < 1$. Hence find $\tanh^{-1}\left(\frac{1}{5}\right)$. [5 marks]

19. Find the domain of $f(x) = \tanh^{-1}\left(\frac{x}{7}\right)$. [1 mark]

20. Express $\cosh^{-1}\left(\frac{5}{4}\right)$ in terms of $\sinh^{-1}$ and of sech^{-1}. [3 marks]

Key sheet

Q	Answer
1	$\frac{17}{15}$
2	$\frac{72}{37}$
3	neither even nor odd
4	$(-6, -2]$
5	$\frac{25}{7}$
6	$\cosh x = \frac{25}{24},\ \tanh x = \frac{7}{25}$
7	$-\frac{19}{180}$
8	$-\frac{8}{13}$
9	$\frac{255}{32}$
10	$\frac{12}{35}$
11	$\frac{5}{12}$
12	$\frac{1}{243}$
13	$\frac{3}{2}$
14	$\frac{4097}{1800}$
15	$\frac{185}{72}$
16	$\frac{\log_e \frac{7}{5}}{2}$
17	$\log_e 5$
18	$\frac{\log_e \frac{3}{2}}{2}$
19	$(-7, 7)$
20	$\sinh^{-1}\left(\frac{3}{4}\right) = \operatorname{sech}^{-1}\left(\frac{4}{5}\right)$

Solutions

1. Find $\cosh(\ln \frac{5}{3})$. [1]

$\cosh x = \frac{e^x + e^{-x}}{2}$

$\cosh(\ln \frac{5}{3}) = \frac{\frac{5}{3} + \frac{3}{5}}{2} = \frac{17}{15}$

Ans: $\frac{17}{15}$

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2. Find $\cosh^2(\ln \frac{5}{2}) - \sinh^2(\ln \frac{5}{2}) + \tanh(\ln 6)$. [2]

$\cosh^2 x - \sinh^2 x = 1$

$\tanh(\ln 6) = \frac{6 - \frac{1}{6}}{6 + \frac{1}{6}} = \frac{35}{37}$

$= 1 + \frac{35}{37} = \frac{72}{37}$

Ans: $\frac{72}{37}$

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3. The function $f(x) = \cosh^4 x + x^6 \sinh 3x$ is [1]

$\cosh$ and sech are even; $\sinh$ and $\tanh$ are odd.

Replacing x by $-x$: one term keeps its sign and the other changes, so f is neither.

Ans: neither even nor odd

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4. Find the range of $f(x) = 4 \operatorname{sech} x - 6$. [2]

$0 < \operatorname{sech} x \leq 1$.

$-6 < f(x) \leq -2$

Range $= (-6, -2]$

Ans: $(-6, -2]$

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5. If $\tanh x = \frac{7}{25}$, find $\coth x$. [1]

$\coth x = \frac{1}{\tanh x} = \frac{25}{7}$

Ans: $\frac{25}{7}$

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6. If $\sinh x = \frac{7}{24}$, find $\cosh x$ and $\tanh x$. [2]

$\cosh^2 x = 1 + \sinh^2 x = 1 + \frac{49}{576} = \frac{625}{576}$

Since $\cosh x \geq 1$, $\cosh x = \frac{25}{24}$

$\tanh x = \frac{\sinh x}{\cosh x} = \frac{\frac{7}{24}}{\frac{25}{24}} = \frac{7}{25}$

Ans: $\cosh x = \frac{25}{24}$, $\tanh x = \frac{7}{25}$

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7. If $\sinh x = \frac{5}{12}$ and $\cosh y = \frac{17}{15}$, with $x, y > 0$, find $\sinh(x - y)$. [3]

$\cosh x = \sqrt{1 + \frac{25}{144}} = \frac{13}{12}$, $\sinh y = \sqrt{\frac{289}{225} - 1} = \frac{8}{15}$

$\sinh(x - y) = \sinh x \cosh y - \cosh x \sinh y$

$= \frac{5}{12} \cdot \frac{17}{15} - \frac{13}{12} \cdot \frac{8}{15} = -\frac{19}{180}$

Ans: $-\frac{19}{180}$

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8. Prove that $\tanh(x-y)=\frac{\tanh x-\tanh y}{1-\tanh x\tanh y}$. Hence find $\tanh(x-y)$ when $\tanh x=\frac{1}{4}$ and $\tanh y=\frac{3}{4}$. [5]

$\tanh(x-y)=\frac{\sinh x\cosh y-\cosh x\sinh y}{\cosh x\cosh y-\sinh x\sinh y}$

Divide the numerator and the denominator by $\cosh x\cosh y$:

$=\frac{\tanh x-\tanh y}{1-\tanh x\tanh y}$

Hence:

$\tanh(x-y)=\frac{\frac{1}{4}-\frac{3}{4}}{1-\frac{3}{16}}=-\frac{8}{13}$

Ans: $-\frac{8}{13}$

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9. If $\sinh x=\frac{15}{8}$ and $\cosh x=\frac{17}{8}$, find $\sinh 2x$. [1]

$\sinh 2x=2\sinh x\cosh x=2\cdot\frac{15}{8}\cdot\frac{17}{8}=\frac{255}{32}$

Ans: $\frac{255}{32}$

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10. If $\tanh x=\frac{1}{6}$, find $\sinh 2x$. [2]

$\sinh 2x=\frac{2\tanh x}{1-\tanh^2 x}=\frac{\frac{1}{3}}{1-\frac{1}{36}}=\frac{12}{35}$

Ans: $\frac{12}{35}$

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11. Prove that $\sinh 2x=\frac{2\tanh x}{1-\tanh^2 x}$. Hence find $\sinh 2x$ when $\tanh x=\frac{1}{5}$. [5]

$\sinh 2x=2\sinh x\cosh x=\frac{2\sinh x\cosh x}{\cosh^2 x-\sinh^2 x}$ (the denominator is 1)

Divide the numerator and the denominator by $\cosh^2 x$:

$=\frac{2\tanh x}{1-\tanh^2 x}$

Hence $\sinh 2x=\frac{5}{12}$

Ans: $\frac{5}{12}$

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12. Find $(\cosh x+\sinh x)^5$ when $x=\ln\frac{1}{3}$. [1]

$(\cosh x+\sinh x)^5=e^{5x}$

$=e^{5\ln\frac{1}{3}}=\left(\frac{1}{3}\right)^5=\frac{1}{243}$

Ans: $\frac{1}{243}$

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13. If $\tanh x=\frac{1}{5}$, find $\cosh 2x+\sinh 2x$. [3]

$\cosh 2nx+\sinh 2nx=\left(\frac{1+\tanh x}{1-\tanh x}\right)^n$ with $n=1$

$\frac{1+\frac{1}{5}}{1-\frac{1}{5}}=\frac{3}{2}$

$\text{Answer}=\left(\frac{3}{2}\right)^1=\frac{3}{2}$

Ans: $\frac{3}{2}$

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14. If $\sinh x=\frac{21}{20}$ and $\cosh y=\frac{13}{12}$, find $\cosh(x+y)\cosh(x-y)$. [2]

$\cosh(x+y)\cosh(x-y)=\sinh^2 x+\cosh^2 y$

$=\frac{441}{400}+\frac{169}{144}=\frac{4097}{1800}$

Ans: $\frac{4097}{1800}$

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15. Prove that $\sinh(x+y)-\sinh(x-y)=2\cosh x\sinh y$. Hence find $\sinh(x+y)-\sinh(x-y)$ when $\cosh x=\frac{37}{12}$ and $\sinh y=\frac{5}{12}$ $(x,y>0)$. [5]

Subtract the formula for $\sinh(x-y)$ from that for $\sinh(x+y)$: the $\sinh x\cosh y$ terms cancel.

So the left side equals $2\cosh x\sinh y$.

Hence $2\cosh x\sinh y=2\cdot\frac{37}{12}\cdot\frac{5}{12}=\frac{185}{72}$

Ans: $\frac{185}{72}$

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16. Find $\coth^{-1}(6)$. [2]

$\coth^{-1}x=\frac{1}{2}\log_e\frac{x+1}{x-1}$

$=\frac{1}{2}\log_e\frac{7}{5}=\frac{\log_e\frac{7}{5}}{2}$

Ans: $\frac{\log_e\frac{7}{5}}{2}$

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17. Find $\operatorname{cosech}^{-1}\left(\frac{5}{12}\right)$. [3]

$\operatorname{cosech}^{-1}x=\log_e\frac{1+\sqrt{1+x^2}}{x}$ (for $x>0$)

$\sqrt{1+\frac{25}{144}}=\frac{13}{12}$

$=\log_e 5$

Ans: $\log_e 5$

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18. Prove that $\tanh^{-1}x=\frac{1}{2}\log_e\frac{1+x}{1-x}$, $|x|<1$. Hence find $\tanh^{-1}\left(\frac{1}{5}\right)$. [5]

Let $y=\tanh^{-1}x$, so $x=\frac{e^y-e^{-y}}{e^y+e^{-y}}=\frac{e^{2y}-1}{e^{2y}+1}$.

$xe^{2y}+x=e^{2y}-1\Rightarrow e^{2y}(1-x)=1+x$.

$e^{2y}=\frac{1+x}{1-x}\Rightarrow y=\frac{1}{2}\log_e\frac{1+x}{1-x}$.

Hence $\tanh^{-1}\left(\frac{1}{5}\right)=\frac{\log_e\frac{3}{2}}{2}$

Ans: $\frac{\log_e\frac{3}{2}}{2}$

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19. Find the domain of $f(x)=\tanh^{-1}\left(\frac{x}{7}\right)$. [1]

$\tanh^{-1}u$ needs $-1<u<1$.

$-1<\frac{x}{7}<1\Rightarrow -7<x<7$

Ans: $(-7,7)$

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20. Express $\cosh^{-1}\left(\frac{5}{4}\right)$ in terms of $\sinh^{-1}$ and of sech^{-1}. [3]

$\cosh^{-1}x=\sinh^{-1}\sqrt{x^2-1}=\operatorname{sech}^{-1}\frac{1}{x}$ for $x\geq 1$.

$\sqrt{\frac{25}{16}-1}=\frac{3}{4}$, $\frac{1}{\frac{5}{4}}=\frac{4}{5}$

$\cosh^{-1}\left(\frac{5}{4}\right)=\sinh^{-1}\left(\frac{3}{4}\right)=\operatorname{sech}^{-1}\left(\frac{4}{5}\right)$

Ans: $\sinh^{-1}\left(\frac{3}{4}\right)=\operatorname{sech}^{-1}\left(\frac{4}{5}\right)$
