EXPRESSWALA ACADEMY

Maths 1A: Hyperbolic Functions

Question Paper, Set 04

Time: 1 hour 30 minutes | Maximum marks: 50

Name: _____ Class: _____ Date: _____

1. Answer all 20 questions. The marks for each question are shown in brackets.
2. Show your working for the 2, 3 and 5 mark questions.
3. The key sheet and worked solutions are at the end of this paper.

1. Find $\cosh(\ln 3)$. [1 mark]

2. Find $\cosh(\ln 2) + \sinh(\ln 6)$. [2 marks]

3. The function $f(x) = x^2 \cosh 5x + x \sinh 5x$ is [1 mark]

4. Find the range of $f(x) = 6\tanh x - 2$. [2 marks]

5. If $\sinh x = \frac{8}{15}$, find $\operatorname{cosech} x$. [1 mark]

6. If $\tanh x = \frac{12}{13}$, find $\operatorname{sech} x$. [2 marks]

7. If $\sinh x = \frac{12}{5}$ and $\cosh y = \frac{17}{15}$, with $x, y > 0$, find $\sinh(x - y)$. [3 marks]

8. Prove that $\sinh(x - y) = \sinh x \cosh y - \cosh x \sinh y$. Hence find $\sinh(x - y)$ when $\sinh x = \frac{12}{5}$ and $\cosh y = \frac{29}{21}$ $(x, y > 0)$. [5 marks]

9. If $\sinh x = \frac{8}{15}$, find $\cosh 2x$. [1 mark]

10. If $\tanh x = \frac{1}{3}$, find $\sinh 2x$. [2 marks]

11. Prove that $\sinh 3x = 3\sinh x + 4\sinh^3 x$. Hence find $\sinh 3x$ when $\sinh x = \frac{2}{3}$. [5 marks]

12. Find $(\cosh x - \sinh x)^3$ when $x = \ln \frac{3}{2}$. [1 mark]

13. If $\tanh x = \frac{2}{7}$, find $\cosh 6x + \sinh 6x$. [3 marks]

14. If $\sinh x = \frac{12}{5}$ and $\cosh y = \frac{29}{20}$, find $\cosh(x + y)\cosh(x - y)$. [2 marks]

15. Prove that $\cosh(x + y) + \cosh(x - y) = 2\cosh x \cosh y$. Hence find $\cosh(x + y) + \cosh(x - y)$ when $\cosh x = \frac{29}{21}$ and $\cosh y = \frac{17}{8}$ $(x, y > 0)$. [5 marks]

16. Find $\coth^{-1}(5)$. [2 marks]

17. Find $\cosh^{-1}\left(\frac{13}{5}\right)$. [3 marks]

18. Prove that $\sinh^{-1} x = \log_e\left(x + \sqrt{x^2 + 1}\right)$. Hence find $\sinh^{-1}\left(\frac{7}{24}\right)$. [5 marks]

19. Find the domain of $f(x) = \operatorname{sech}^{-1}\left(\frac{x}{4}\right)$. [1 mark]

20. Express $\sinh^{-1}\left(\frac{15}{8}\right)$ in terms of $\tanh^{-1}$ and of $\cosh^{-1}$. [3 marks]

Key sheet

Q	Answer
1	$\frac{5}{3}$
2	$\frac{25}{6}$
3	even
4	$(-8, 4)$
5	$\frac{15}{8}$
6	$\frac{5}{13}$
7	$\frac{4}{3}$
8	$\frac{88}{105}$
9	$\frac{353}{225}$
10	$\frac{3}{4}$
11	$\frac{86}{27}$
12	$\frac{8}{27}$
13	$\frac{729}{125}$
14	$\frac{629}{80}$
15	$\frac{493}{84}$
16	$\frac{\log_e \frac{3}{2}}{2}$
17	$\log_e 5$
18	$\log_e \frac{4}{3}$
19	$(0, 4]$
20	$\tanh^{-1}\left(\frac{15}{17}\right) = \cosh^{-1}\left(\frac{17}{8}\right)$

Solutions

1. Find $\cosh(\ln 3)$. [1]

$\cosh x = \frac{e^x + e^{-x}}{2}$

$\cosh(\ln 3) = \frac{3 + \frac{1}{3}}{2} = \frac{5}{3}$

Ans: $\frac{5}{3}$

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2. Find $\cosh(\ln 2) + \sinh(\ln 6)$. [2]

$\cosh(\ln 2) = \frac{2 + \frac{1}{2}}{2} = \frac{5}{4}$

$\sinh(\ln 6) = \frac{6 - \frac{1}{6}}{2} = \frac{35}{12}$

$\text{Sum} = \frac{25}{6}$

Ans: $\frac{25}{6}$

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3. The function $f(x) = x^2 \cosh 5x + x \sinh 5x$ is [1]

$\cosh$ and sech are even; $\sinh$ and $\tanh$ are odd.

Replacing x by $-x$: $f(-x) = f(x)$, so f is even.

Ans: even

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4. Find the range of $f(x) = 6 \tanh x - 2$. [2]

$-1 < \tanh x < 1$.

$-2 - 6 < f(x) < -2 + 6$

Range $= (-8, 4)$

Ans: $(-8, 4)$

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5. If $\sinh x = \frac{8}{15}$, find $\text{cosech}\, x$. [1]

$\text{cosech}\, x = \frac{1}{\sinh x} = \frac{15}{8}$

Ans: $\frac{15}{8}$

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6. If $\tanh x = \frac{12}{13}$, find $\text{sech}\, x$. [2]

$\text{sech}^2 x = 1 - \tanh^2 x = 1 - \frac{144}{169} = \frac{25}{169}$

$\text{sech}\, x > 0$, so $\text{sech}\, x = \frac{5}{13}$

Ans: $\frac{5}{13}$

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7. If $\sinh x = \frac{12}{5}$ and $\cosh y = \frac{17}{15}$, with $x, y > 0$, find $\sinh(x - y)$. [3]

$\cosh x = \sqrt{1 + \frac{144}{25}} = \frac{13}{5}$, $\sinh y = \sqrt{\frac{289}{225} - 1} = \frac{8}{15}$

$\sinh(x - y) = \sinh x \cosh y - \cosh x \sinh y$

$= \frac{12}{5} \cdot \frac{17}{15} - \frac{13}{5} \cdot \frac{8}{15} = \frac{4}{3}$

Ans: $\frac{4}{3}$

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8. Prove that $\sinh(x-y) = \sinh x \cosh y - \cosh x \sinh y$. Hence find $\sinh(x-y)$ when $\sinh x = \frac{12}{5}$ and $\cosh y = \frac{29}{21}$ $(x, y > 0)$. [5]

$\text{RHS} = \frac{(e^x - e^{-x})(e^y + e^{-y}) - (e^x + e^{-x})(e^y - e^{-y})}{4}$

$= \frac{(e^{x+y} + e^{x-y} - e^{-x+y} - e^{-x-y}) - (e^{x+y} - e^{x-y} + e^{-x+y} - e^{-x-y})}{4}$

$= \frac{2e^{x-y} - 2e^{-(x-y)}}{4} = \sinh(x-y) = \text{LHS}$

Hence:

$\cosh x = \frac{13}{5},\ \sinh y = \frac{20}{21}$

$\sinh(x-y) = \frac{12}{5} \cdot \frac{29}{21} - \frac{13}{5} \cdot \frac{20}{21} = \frac{88}{105}$

Ans: $\frac{88}{105}$

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9. If $\sinh x = \frac{8}{15}$, find $\cosh 2x$. [1]

$\cosh 2x = 1 + 2\sinh^2 x = 1 + 2 \cdot \frac{64}{225} = \frac{353}{225}$

Ans: $\frac{353}{225}$

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10. If $\tanh x = \frac{1}{3}$, find $\sinh 2x$. [2]

$\sinh 2x = \frac{2\tanh x}{1 - \tanh^2 x} = \frac{\frac{2}{3}}{1 - \frac{1}{9}} = \frac{3}{4}$

Ans: $\frac{3}{4}$

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11. Prove that $\sinh 3x = 3\sinh x + 4\sinh^3 x$. Hence find $\sinh 3x$ when $\sinh x = \frac{2}{3}$. [5]

$\sinh 3x = \sinh(2x + x) = \sinh 2x \cosh x + \cosh 2x \sinh x$

$= 2\sinh x \cosh^2 x + (1 + 2\sinh^2 x)\sinh x$

$= 2\sinh x(1 + \sinh^2 x) + \sinh x + 2\sinh^3 x$

$= 3\sinh x + 4\sinh^3 x$

Hence $\sinh 3x = 3(\frac{2}{3}) + 4(\frac{2}{3})^3 = \frac{86}{27}$

Ans: $\frac{86}{27}$

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12. Find $(\cosh x - \sinh x)^3$ when $x = \ln \frac{3}{2}$. [1]

$(\cosh x - \sinh x)^3 = e^{-3x}$

$= e^{-3\ln\frac{3}{2}} = \left(\frac{3}{2}\right)^{-3} = \frac{8}{27}$

Ans: $\frac{8}{27}$

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13. If $\tanh x = \frac{2}{7}$, find $\cosh 6x + \sinh 6x$. [3]

$\cosh 2nx + \sinh 2nx = \left(\frac{1+\tanh x}{1-\tanh x}\right)^n$ with $n = 3$

$\frac{1+\frac{2}{7}}{1-\frac{2}{7}} = \frac{9}{5}$

$\text{Answer} = \left(\frac{9}{5}\right)^3 = \frac{729}{125}$

Ans: $\frac{729}{125}$

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14. If $\sinh x = \frac{12}{5}$ and $\cosh y = \frac{29}{20}$, find $\cosh(x+y)\cosh(x-y)$. [2]

$\cosh(x+y)\cosh(x-y) = \sinh^2 x + \cosh^2 y$

$= \frac{144}{25} + \frac{841}{400} = \frac{629}{80}$

Ans: $\frac{629}{80}$

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15. Prove that $\cosh(x+y)+\cosh(x-y)=2\cosh x\cosh y$. Hence find $\cosh(x+y)+\cosh(x-y)$ when $\cosh x=\frac{29}{21}$ and $\cosh y=\frac{17}{8}$ $(x,y>0)$. [5]

Add the formulas for $\cosh(x+y)$ and $\cosh(x-y)$: the $\sinh x\sinh y$ terms cancel.

So the left side equals $2\cosh x\cosh y$.

Hence $2\cosh x\cosh y=2\cdot\frac{29}{21}\cdot\frac{17}{8}=\frac{493}{84}$

Ans: $\frac{493}{84}$

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16. Find $\coth^{-1}(5)$. [2]

$\coth^{-1}x=\frac{1}{2}\log_e\frac{x+1}{x-1}$

$=\frac{1}{2}\log_e\frac{6}{4}=\frac{\log_e\frac{3}{2}}{2}$

Ans: $\frac{\log_e\frac{3}{2}}{2}$

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17. Find $\cosh^{-1}\left(\frac{13}{5}\right)$. [3]

$\cosh^{-1}x=\log_e\left(x+\sqrt{x^2-1}\right)$

$\sqrt{\frac{169}{25}-1}=\frac{12}{5}$

$\cosh^{-1}\left(\frac{13}{5}\right)=\log_e\left(\frac{13}{5}+\frac{12}{5}\right)=\log_e 5$

Ans: $\log_e 5$

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18. Prove that $\sinh^{-1}x=\log_e\left(x+\sqrt{x^2+1}\right)$. Hence find $\sinh^{-1}\left(\frac{7}{24}\right)$. [5]

Let $y=\sinh^{-1}x$, so $x=\sinh y=\frac{e^y-e^{-y}}{2}$.

Multiply by $2e^y$: $e^{2y}-2xe^y-1=0$, a quadratic in e^y.

$e^y=\frac{2x\pm\sqrt{4x^2+4}}{2}=x\pm\sqrt{x^2+1}$.

$e^y>0$, so $e^y=x+\sqrt{x^2+1}$ and $y=\log_e\left(x+\sqrt{x^2+1}\right)$.

Hence $\sinh^{-1}\left(\frac{7}{24}\right)=\log_e\frac{4}{3}$

Ans: $\log_e\frac{4}{3}$

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19. Find the domain of $f(x)=\operatorname{sech}^{-1}\left(\frac{x}{4}\right)$. [1]

$\operatorname{sech}^{-1}u$ needs $0<u\leq 1$.

$0<\frac{x}{4}\leq 1\Rightarrow 0<x\leq 4$

Ans: $(0,4]$

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20. Express $\sinh^{-1}\left(\frac{15}{8}\right)$ in terms of $\tanh^{-1}$ and of $\cosh^{-1}$. [3]

$\sinh^{-1}x=\tanh^{-1}\frac{x}{\sqrt{1+x^2}}=\cosh^{-1}\sqrt{x^2+1}$ for $x>0$.

$\sqrt{1+\frac{225}{64}}=\frac{17}{8}$

$\tanh^{-1}\frac{\frac{15}{8}}{\frac{17}{8}}=\tanh^{-1}\left(\frac{15}{17}\right)$

$\sinh^{-1}\left(\frac{15}{8}\right)=\tanh^{-1}\left(\frac{15}{17}\right)=\cosh^{-1}\left(\frac{17}{8}\right)$

Ans: $\tanh^{-1}\left(\frac{15}{17}\right)=\cosh^{-1}\left(\frac{17}{8}\right)$
