EXPRESSWALA ACADEMY

Maths 1A: Hyperbolic Functions

Question Paper, Set 03

Time: 1 hour 30 minutes | Maximum marks: 50

Name: _____ Class: _____ Date: _____

1. Answer all 20 questions. The marks for each question are shown in brackets.
2. Show your working for the 2, 3 and 5 mark questions.
3. The key sheet and worked solutions are at the end of this paper.

1. Find $\tanh(\ln 5)$. [1 mark]

2. Find $\operatorname{sech}(\ln \frac{3}{2})$. [2 marks]

3. The function $f(x) = x\cosh 5x + \tanh 5x$ is [1 mark]

4. Find the range of $f(x) = 2\tanh x - 5$. [2 marks]

5. If $\tanh x = \frac{15}{17}$, find $\coth x$. [1 mark]

6. If $\cosh x = \frac{5}{3}$ and x is negative, find $\sinh x$. [2 marks]

7. If $\tanh x = \frac{1}{4}$ and $\tanh y = \frac{2}{5}$, find $\tanh(x+y)$. [3 marks]

8. Prove that $\sinh(x-y) = \sinh x\cosh y - \cosh x\sinh y$. Hence find $\sinh(x-y)$ when $\sinh x = \frac{5}{12}$ and $\cosh y = \frac{5}{3}$ $(x, y > 0)$. [5 marks]

9. If $\cosh x = \frac{5}{4}$, find $\cosh 2x$. [1 mark]

10. If $\cosh x = \frac{5}{3}$, find $\cosh 3x$. [2 marks]

11. Prove that $\tanh 2x = \frac{2\tanh x}{1+\tanh^2 x}$. Hence find $\tanh 2x$ when $\tanh x = \frac{1}{2}$. [5 marks]

12. Find $(\cosh x + \sinh x)^4$ when $x = \ln 2$. [1 mark]

13. If $\tanh x = \frac{1}{6}$, find $\cosh 6x + \sinh 6x$. [3 marks]

14. If $\sinh x = \frac{20}{21}$ and $\sinh y = \frac{4}{3}$, find $\sinh(x+y)\sinh(x-y)$. [2 marks]

15. Prove that $\cosh(x+y) + \cosh(x-y) = 2\cosh x\cosh y$. Hence find $\cosh(x+y) + \cosh(x-y)$ when $\cosh x = \frac{25}{24}$ and $\cosh y = \frac{13}{5}$ $(x, y > 0)$. [5 marks]

16. Find $\coth^{-1}(7)$. [2 marks]

17. Find $\operatorname{cosech}^{-1}\left(-\frac{15}{8}\right)$. [3 marks]

18. Prove that $\sinh^{-1} x = \log_e\left(x + \sqrt{x^2+1}\right)$. Hence find $\sinh^{-1}\left(\frac{21}{20}\right)$. [5 marks]

19. Find the domain of $f(x) = \operatorname{sech}^{-1}\left(\frac{x}{8}\right)$. [1 mark]

20. Express $\cosh^{-1}\left(\frac{25}{24}\right)$ in terms of $\sinh^{-1}$ and of sech^{-1}. [3 marks]

Key sheet

Q	Answer
1	$\frac{12}{13}$
2	$\frac{12}{13}$
3	odd
4	$(-7, -3)$
5	$\frac{17}{15}$
6	$-\frac{4}{3}$
7	$\frac{13}{22}$
8	$-\frac{3}{4}$
9	$\frac{17}{8}$
10	$\frac{365}{27}$
11	$\frac{4}{5}$
12	16
13	$\frac{343}{125}$
14	$-\frac{128}{147}$
15	$\frac{65}{12}$
16	$\frac{\log_e \frac{4}{3}}{2}$
17	$-\log_e \frac{5}{3}$
18	$\log_e \frac{5}{2}$
19	$(0, 8]$
20	$\sinh^{-1}\left(\frac{7}{24}\right) = \operatorname{sech}^{-1}\left(\frac{24}{25}\right)$

Solutions

1. Find $\tanh(\ln 5)$. [1]

$\tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}}$

$\tanh(\ln 5) = \frac{5 - \frac{1}{5}}{5 + \frac{1}{5}} = \frac{12}{13}$

Ans: $\frac{12}{13}$

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2. Find $\operatorname{sech}(\ln \frac{3}{2})$. [2]

$\operatorname{sech} x = \frac{2}{e^x + e^{-x}}$

$e^{\ln \frac{3}{2}} = \frac{3}{2},\ e^{-\ln \frac{3}{2}} = \frac{2}{3}$

$\operatorname{sech}(\ln \frac{3}{2}) = \frac{12}{13}$

Ans: $\frac{12}{13}$

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3. The function $f(x) = x \cosh 5x + \tanh 5x$ is [1]

$\cosh$ and sech are even; $\sinh$ and $\tanh$ are odd.

Replacing x by $-x$: $f(-x) = -f(x)$, so f is odd.

Ans: odd

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4. Find the range of $f(x) = 2\tanh x - 5$. [2]

$-1 < \tanh x < 1$.

$-5 - 2 < f(x) < -5 + 2$

Range $= (-7, -3)$

Ans: $(-7, -3)$

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5. If $\tanh x = \frac{15}{17}$, find $\coth x$. [1]

$\coth x = \frac{1}{\tanh x} = \frac{17}{15}$

Ans: $\frac{17}{15}$

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6. If $\cosh x = \frac{5}{3}$ and x is negative, find $\sinh x$. [2]

$\sinh^2 x = \cosh^2 x - 1 = \frac{25}{9} - 1 = \frac{16}{9}$

x is negative, so $\sinh x$ is negative: $\sinh x = -\frac{4}{3}$

Ans: $-\frac{4}{3}$

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7. If $\tanh x = \frac{1}{4}$ and $\tanh y = \frac{2}{5}$, find $\tanh(x + y)$. [3]

$\tanh(x + y) = \frac{\tanh x + \tanh y}{1 + \tanh x \tanh y}$

$= \frac{\frac{1}{4} + \frac{2}{5}}{1 + \frac{1}{10}} = \frac{13}{22}$

Ans: $\frac{13}{22}$

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8. Prove that $\sinh(x-y) = \sinh x \cosh y - \cosh x \sinh y$. Hence find $\sinh(x-y)$ when $\sinh x = \frac{5}{12}$ and $\cosh y = \frac{5}{3}$ $(x, y > 0)$. [5]

$\text{RHS} = \frac{(e^x - e^{-x})(e^y + e^{-y}) - (e^x + e^{-x})(e^y - e^{-y})}{4}$

$= \frac{(e^{x+y} + e^{x-y} - e^{-x+y} - e^{-x-y}) - (e^{x+y} - e^{x-y} + e^{-x+y} - e^{-x-y})}{4}$

$= \frac{2e^{x-y} - 2e^{-(x-y)}}{4} = \sinh(x-y) = \text{LHS}$

Hence:

$\cosh x = \frac{13}{12},\ \sinh y = \frac{4}{3}$

$\sinh(x-y) = \frac{5}{12} \cdot \frac{5}{3} - \frac{13}{12} \cdot \frac{4}{3} = -\frac{3}{4}$

Ans: $-\frac{3}{4}$

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9. If $\cosh x = \frac{5}{4}$, find $\cosh 2x$. [1]

$\cosh 2x = 2\cosh^2 x - 1 = 2 \cdot \frac{25}{16} - 1 = \frac{17}{8}$

Ans: $\frac{17}{8}$

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10. If $\cosh x = \frac{5}{3}$, find $\cosh 3x$. [2]

$\cosh 3x = 4\cosh^3 x - 3\cosh x$

$= 4(\frac{5}{3})^3 - 3(\frac{5}{3}) = \frac{365}{27}$

Ans: $\frac{365}{27}$

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11. Prove that $\tanh 2x = \frac{2\tanh x}{1+\tanh^2 x}$. Hence find $\tanh 2x$ when $\tanh x = \frac{1}{2}$. [5]

$\tanh 2x = \frac{\sinh 2x}{\cosh 2x} = \frac{2\sinh x \cosh x}{\cosh^2 x + \sinh^2 x}$

Divide the numerator and the denominator by $\cosh^2 x$:

$= \frac{2\tanh x}{1+\tanh^2 x}$

Hence $\tanh 2x = \frac{4}{5}$

Ans: $\frac{4}{5}$

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12. Find $(\cosh x + \sinh x)^4$ when $x = \ln 2$. [1]

$(\cosh x + \sinh x)^4 = e^{4x}$

$= e^{4\ln 2} = 2^4 = 16$

Ans: 16

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13. If $\tanh x = \frac{1}{6}$, find $\cosh 6x + \sinh 6x$. [3]

$\cosh 2nx + \sinh 2nx = \left(\frac{1+\tanh x}{1-\tanh x}\right)^n$ with $n = 3$

$\frac{1+\frac{1}{6}}{1-\frac{1}{6}} = \frac{7}{5}$

$\text{Answer} = \left(\frac{7}{5}\right)^3 = \frac{343}{125}$

Ans: $\frac{343}{125}$

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14. If $\sinh x = \frac{20}{21}$ and $\sinh y = \frac{4}{3}$, find $\sinh(x+y)\sinh(x-y)$. [2]

$\sinh(x+y)\sinh(x-y) = \sinh^2 x - \sinh^2 y$

$= \frac{400}{441} - \frac{16}{9} = -\frac{128}{147}$

Ans: $-\frac{128}{147}$

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15. Prove that $\cosh(x+y)+\cosh(x-y)=2\cosh x\cosh y$. Hence find $\cosh(x+y)+\cosh(x-y)$ when $\cosh x=\frac{25}{24}$ and $\cosh y=\frac{13}{5}$ $(x,y>0)$. [5]

Add the formulas for $\cosh(x+y)$ and $\cosh(x-y)$: the $\sinh x\sinh y$ terms cancel.

So the left side equals $2\cosh x\cosh y$.

Hence $2\cosh x\cosh y=2\cdot\frac{25}{24}\cdot\frac{13}{5}=\frac{65}{12}$

Ans: $\frac{65}{12}$

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16. Find $\coth^{-1}(7)$. [2]

$\coth^{-1}x=\frac{1}{2}\log_e\frac{x+1}{x-1}$

$=\frac{1}{2}\log_e\frac{8}{6}=\frac{\log_e\frac{4}{3}}{2}$

Ans: $\frac{\log_e\frac{4}{3}}{2}$

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17. Find $\operatorname{cosech}^{-1}\left(-\frac{15}{8}\right)$. [3]

$\operatorname{cosech}^{-1}x=\log_e\frac{1-\sqrt{1+x^2}}{x}$ (for $x<0$)

$\sqrt{1+\frac{225}{64}}=\frac{17}{8}$

$=-\log_e\frac{5}{3}$

Ans: $-\log_e\frac{5}{3}$

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18. Prove that $\sinh^{-1}x=\log_e\left(x+\sqrt{x^2+1}\right)$. Hence find $\sinh^{-1}\left(\frac{21}{20}\right)$. [5]

Let $y=\sinh^{-1}x$, so $x=\sinh y=\frac{e^y-e^{-y}}{2}$.

Multiply by $2e^y$: $e^{2y}-2xe^y-1=0$, a quadratic in e^y.

$e^y=\frac{2x\pm\sqrt{4x^2+4}}{2}=x\pm\sqrt{x^2+1}$.

$e^y>0$, so $e^y=x+\sqrt{x^2+1}$ and $y=\log_e\left(x+\sqrt{x^2+1}\right)$.

Hence $\sinh^{-1}\left(\frac{21}{20}\right)=\log_e\frac{5}{2}$

Ans: $\log_e\frac{5}{2}$

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19. Find the domain of $f(x)=\operatorname{sech}^{-1}\left(\frac{x}{8}\right)$. [1]

$\operatorname{sech}^{-1}u$ needs $0<u\leq 1$.

$0<\frac{x}{8}\leq 1\Rightarrow 0<x\leq 8$

Ans: $(0,8]$

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20. Express $\cosh^{-1}\left(\frac{25}{24}\right)$ in terms of $\sinh^{-1}$ and of sech^{-1}. [3]

$\cosh^{-1}x=\sinh^{-1}\sqrt{x^2-1}=\operatorname{sech}^{-1}\frac{1}{x}$ for $x\geq 1$.

$\sqrt{\frac{625}{576}-1}=\frac{7}{24}$, $\frac{1}{\frac{25}{24}}=\frac{24}{25}$

$\cosh^{-1}\left(\frac{25}{24}\right)=\sinh^{-1}\left(\frac{7}{24}\right)=\operatorname{sech}^{-1}\left(\frac{24}{25}\right)$

Ans: $\sinh^{-1}\left(\frac{7}{24}\right)=\operatorname{sech}^{-1}\left(\frac{24}{25}\right)$
