EXPRESSWALA ACADEMY

Maths 1A: Hyperbolic Functions

Question Paper, Set 02

Time: 1 hour 30 minutes | Maximum marks: 50

Name: _____ Class: _____ Date: _____

1. Answer all 20 questions. The marks for each question are shown in brackets.
2. Show your working for the 2, 3 and 5 mark questions.
3. The key sheet and worked solutions are at the end of this paper.

1. Find $\tanh(\ln 3)$. [1 mark]

2. Find $\cosh^2(\ln \frac{7}{3}) - \sinh^2(\ln \frac{7}{3}) + \tanh(\ln \frac{3}{2})$. [2 marks]

3. The function $f(x) = x^6 \cosh 4x + \sinh^7 x$ is [1 mark]

4. Find the range of $f(x) = 6\,\mathrm{sech}\, x + 2$. [2 marks]

5. Simplify $\cosh x + \sinh x$ when $x = \ln 5$. [1 mark]

6. If $\tanh x = \frac{20}{29}$, find $\mathrm{sech}\, x$. [2 marks]

7. If $\cosh x = \frac{17}{8}$ and $\sinh y = \frac{8}{15}$, with $x, y > 0$, find $\cosh(x - y)$. [3 marks]

8. Prove that $\cosh(x + y) = \cosh x \cosh y + \sinh x \sinh y$. Hence find $\cosh(x + y)$ when $\sinh x = \frac{12}{5}$ and $\sinh y = \frac{5}{12}$. [5 marks]

9. If $\sinh x = \frac{5}{12}$, find $\cosh 2x$. [1 mark]

10. If $\tanh x = \frac{3}{7}$, find $\sinh 2x$. [2 marks]

11. Prove that $\cosh 3x = 4\cosh^3 x - 3\cosh x$. Hence find $\cosh 3x$ when $\cosh x = 3$. [5 marks]

12. Find $(\cosh x + \sinh x)^4$ when $x = \ln 3$. [1 mark]

13. If $\tanh x = \frac{3}{5}$, find $\cosh 4x + \sinh 4x$. [3 marks]

14. If $\sinh x = \frac{7}{24}$ and $\sinh y = \frac{20}{21}$, find $\sinh(x + y)\sinh(x - y)$. [2 marks]

15. Prove that $\cosh(x + y) - \cosh(x - y) = 2\sinh x \sinh y$. Hence find $\cosh(x + y) - \cosh(x - y)$ when $\sinh x = \frac{7}{24}$ and $\sinh y = \frac{21}{20}$ $(x, y > 0)$. [5 marks]

16. Find $\tanh^{-1}\left(\frac{3}{4}\right)$. [2 marks]

17. Find $\mathrm{sech}^{-1}\left(\frac{3}{5}\right)$. [3 marks]

18. Prove that $\coth^{-1} x = \frac{1}{2}\log_e \frac{x+1}{x-1}$, $|x| > 1$. Hence find $\coth^{-1}(5)$. [5 marks]

19. Find the domain of $f(x) = \mathrm{sech}^{-1}\left(\frac{x}{7}\right)$. [1 mark]

20. Express $\sinh^{-1}\left(\frac{12}{5}\right)$ in terms of $\tanh^{-1}$ and of $\cosh^{-1}$. [3 marks]

Key sheet

Q	Answer
1	$\frac{4}{5}$
2	$\frac{18}{13}$
3	neither even nor odd
4	$(2, 8]$
5	5
6	$\frac{21}{29}$
7	$\frac{169}{120}$
8	$\frac{229}{60}$
9	$\frac{97}{72}$
10	$\frac{21}{20}$
11	99
12	81
13	16
14	$-\frac{7733}{9408}$
15	$\frac{49}{80}$
16	$\frac{\log_e 7}{2}$
17	$\log_e 3$
18	$\frac{\log_e \frac{3}{2}}{2}$
19	$(0, 7]$
20	$\tanh^{-1}\left(\frac{12}{13}\right) = \cosh^{-1}\left(\frac{13}{5}\right)$

Solutions

1. Find $\tanh(\ln 3)$. [1]

$\tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}}$

$\tanh(\ln 3) = \frac{3 - \frac{1}{3}}{3 + \frac{1}{3}} = \frac{4}{5}$

Ans: $\frac{4}{5}$

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2. Find $\cosh^2(\ln \frac{7}{3}) - \sinh^2(\ln \frac{7}{3}) + \tanh(\ln \frac{3}{2})$. [2]

$\cosh^2 x - \sinh^2 x = 1$

$\tanh(\ln \frac{3}{2}) = \frac{\frac{3}{2} - \frac{2}{3}}{\frac{3}{2} + \frac{2}{3}} = \frac{5}{13}$

$= 1 + \frac{5}{13} = \frac{18}{13}$

Ans: $\frac{18}{13}$

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3. The function $f(x) = x^6 \cosh 4x + \sinh^7 x$ is [1]

$\cosh$ and sech are even; $\sinh$ and $\tanh$ are odd.

Replacing x by $-x$: one term keeps its sign and the other changes, so f is neither.

Ans: neither even nor odd

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4. Find the range of $f(x) = 6 \operatorname{sech} x + 2$. [2]

$0 < \operatorname{sech} x \leq 1$.

$2 < f(x) \leq 8$

Range $= (2, 8]$

Ans: $(2, 8]$

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5. Simplify $\cosh x + \sinh x$ when $x = \ln 5$. [1]

$\cosh x + \sinh x = e^x$

$e^{\ln 5} = 5$

Ans: 5

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6. If $\tanh x = \frac{20}{29}$, find $\operatorname{sech} x$. [2]

$\operatorname{sech}^2 x = 1 - \tanh^2 x = 1 - \frac{400}{841} = \frac{441}{841}$

$\operatorname{sech} x > 0$, so $\operatorname{sech} x = \frac{21}{29}$

Ans: $\frac{21}{29}$

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7. If $\cosh x = \frac{17}{8}$ and $\sinh y = \frac{8}{15}$, with $x, y > 0$, find $\cosh(x - y)$. [3]

$\sinh x = \sqrt{\frac{289}{64} - 1} = \frac{15}{8}$, $\cosh y = \sqrt{1 + \frac{64}{225}} = \frac{17}{15}$

$\cosh(x - y) = \cosh x \cosh y - \sinh x \sinh y$

$= \frac{17}{8} \cdot \frac{17}{15} - \frac{15}{8} \cdot \frac{8}{15} = \frac{169}{120}$

Ans: $\frac{169}{120}$

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8. Prove that $\cosh(x+y) = \cosh x \cosh y + \sinh x \sinh y$. Hence find $\cosh(x+y)$ when $\sinh x = \frac{12}{5}$ and $\sinh y = \frac{5}{12}$. [5]

$\text{RHS} = \frac{(e^x+e^{-x})(e^y+e^{-y})+(e^x-e^{-x})(e^y-e^{-y})}{4}$

$= \frac{(e^{x+y}+e^{x-y}+e^{-x+y}+e^{-x-y})+(e^{x+y}-e^{x-y}-e^{-x+y}+e^{-x-y})}{4}$

$= \frac{2e^{x+y}+2e^{-(x+y)}}{4} = \cosh(x+y) = \text{LHS}$

Hence:

$\cosh x = \frac{13}{5},\ \cosh y = \frac{13}{12}$

$\cosh(x+y) = \frac{13}{5}\cdot\frac{13}{12} + \frac{12}{5}\cdot\frac{5}{12} = \frac{229}{60}$

Ans: $\frac{229}{60}$

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9. If $\sinh x = \frac{5}{12}$, find $\cosh 2x$. [1]

$\cosh 2x = 1 + 2\sinh^2 x = 1 + 2\cdot\frac{25}{144} = \frac{97}{72}$

Ans: $\frac{97}{72}$

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10. If $\tanh x = \frac{3}{7}$, find $\sinh 2x$. [2]

$\sinh 2x = \frac{2\tanh x}{1-\tanh^2 x} = \frac{\frac{6}{7}}{1-\frac{9}{49}} = \frac{21}{20}$

Ans: $\frac{21}{20}$

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11. Prove that $\cosh 3x = 4\cosh^3 x - 3\cosh x$. Hence find $\cosh 3x$ when $\cosh x = 3$. [5]

$\cosh 3x = \cosh(2x+x) = \cosh 2x \cosh x + \sinh 2x \sinh x$

$= (2\cosh^2 x - 1)\cosh x + 2\sinh^2 x \cosh x$

$= 2\cosh^3 x - \cosh x + 2(\cosh^2 x - 1)\cosh x$

$= 4\cosh^3 x - 3\cosh x$

Hence $\cosh 3x = 4(3)^3 - 3(3) = 99$

Ans: 99

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12. Find $(\cosh x + \sinh x)^4$ when $x = \ln 3$. [1]

$(\cosh x + \sinh x)^4 = e^{4x}$

$= e^{4\ln 3} = 3^4 = 81$

Ans: 81

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13. If $\tanh x = \frac{3}{5}$, find $\cosh 4x + \sinh 4x$. [3]

$\cosh 2nx + \sinh 2nx = \left(\frac{1+\tanh x}{1-\tanh x}\right)^n$ with $n = 2$

$\frac{1+\frac{3}{5}}{1-\frac{3}{5}} = 4$

$\text{Answer} = (4)^2 = 16$

Ans: 16

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14. If $\sinh x = \frac{7}{24}$ and $\sinh y = \frac{20}{21}$, find $\sinh(x+y)\sinh(x-y)$. [2]

$\sinh(x+y)\sinh(x-y) = \sinh^2 x - \sinh^2 y$

$= \frac{49}{576} - \frac{400}{441} = -\frac{7733}{9408}$

Ans: $-\frac{7733}{9408}$

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15. Prove that $\cosh(x+y)-\cosh(x-y)=2\sinh x\sinh y$. Hence find $\cosh(x+y)-\cosh(x-y)$ when $\sinh x=\frac{7}{24}$ and $\sinh y=\frac{21}{20}$ $(x,y>0)$. [5]

Subtract the formula for $\cosh(x-y)$ from that for $\cosh(x+y)$: the $\cosh x\cosh y$ terms cancel.

So the left side equals $2\sinh x\sinh y$.

Hence $2\sinh x\sinh y=2\cdot\frac{7}{24}\cdot\frac{21}{20}=\frac{49}{80}$

Ans: $\frac{49}{80}$

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16. Find $\tanh^{-1}\left(\frac{3}{4}\right)$. [2]

$\tanh^{-1}x=\frac{1}{2}\log_e\frac{1+x}{1-x}$

$=\frac{1}{2}\log_e\frac{\frac{7}{4}}{\frac{1}{4}}=\frac{\log_e 7}{2}$

Ans: $\frac{\log_e 7}{2}$

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17. Find $\operatorname{sech}^{-1}\left(\frac{3}{5}\right)$. [3]

$\operatorname{sech}^{-1}x=\log_e\frac{1+\sqrt{1-x^2}}{x}$

$\sqrt{1-\frac{9}{25}}=\frac{4}{5}$

$=\log_e\frac{1+\frac{4}{5}}{\frac{3}{5}}=\log_e 3$

Ans: $\log_e 3$

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18. Prove that $\coth^{-1}x=\frac{1}{2}\log_e\frac{x+1}{x-1},\ |x|>1$. Hence find $\coth^{-1}(5)$. [5]

Let $y=\coth^{-1}x$, so $x=\frac{e^{2y}+1}{e^{2y}-1}$.

$xe^{2y}-x=e^{2y}+1\Rightarrow e^{2y}(x-1)=x+1$.

$e^{2y}=\frac{x+1}{x-1}\Rightarrow y=\frac{1}{2}\log_e\frac{x+1}{x-1}$.

Hence $\coth^{-1}(5)=\frac{\log_e\frac{3}{2}}{2}$

Ans: $\frac{\log_e\frac{3}{2}}{2}$

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19. Find the domain of $f(x)=\operatorname{sech}^{-1}\left(\frac{x}{7}\right)$. [1]

$\operatorname{sech}^{-1}u$ needs $0<u\leq 1$.

$0<\frac{x}{7}\leq 1\Rightarrow 0<x\leq 7$

Ans: $(0,7]$

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20. Express $\sinh^{-1}\left(\frac{12}{5}\right)$ in terms of $\tanh^{-1}$ and of $\cosh^{-1}$. [3]

$\sinh^{-1}x=\tanh^{-1}\frac{x}{\sqrt{1+x^2}}=\cosh^{-1}\sqrt{x^2+1}$ for $x>0$.

$\sqrt{1+\frac{144}{25}}=\frac{13}{5}$

$\tanh^{-1}\frac{\frac{12}{5}}{\frac{13}{5}}=\tanh^{-1}\left(\frac{12}{13}\right)$

$\sinh^{-1}\left(\frac{12}{5}\right)=\tanh^{-1}\left(\frac{12}{13}\right)=\cosh^{-1}\left(\frac{13}{5}\right)$

Ans: $\tanh^{-1}\left(\frac{12}{13}\right)=\cosh^{-1}\left(\frac{13}{5}\right)$
