EXPRESSWALA ACADEMY

Maths 1A: Hyperbolic Functions

Question Paper, Set 10

Time: 1 hour 30 minutes | Maximum marks: 50

Name: _____ Class: _____ Date: _____

1. Answer all 20 questions. The marks for each question are shown in brackets.
2. Show your working for the 2, 3 and 5 mark questions.
3. The key sheet and worked solutions are at the end of this paper.

1. Find $\tanh(\ln 2)$. [1 mark]

2. Find $2\sinh(\ln 5)\cosh(\ln 5)$. [2 marks]

3. The function $f(x) = \sinh^7 x + x\cosh 2x$ is [1 mark]

4. Find the range of $f(x) = 2\cosh x - 1$. [2 marks]

5. If $\cosh x = \frac{5}{4}$, find $\operatorname{sech} x$. [1 mark]

6. If $\cosh x = \frac{29}{20}$ and x is negative, find $\sinh x$. [2 marks]

7. If $\cosh x = \frac{5}{4}$ and $\sinh y = \frac{15}{8}$, with $x, y > 0$, find $\cosh(x - y)$. [3 marks]

8. Prove that $\sinh(x - y) = \sinh x \cosh y - \cosh x \sinh y$. Hence find $\sinh(x - y)$ when $\sinh x = \frac{3}{4}$ and $\cosh y = \frac{5}{3}$ $(x, y > 0)$. [5 marks]

9. If $\cosh x = \frac{13}{12}$, find $\cosh 2x$. [1 mark]

10. If $\tanh x = \frac{1}{5}$, find $\sinh 2x$. [2 marks]

11. Prove that $\tanh 2x = \frac{2\tanh x}{1+\tanh^2 x}$. Hence find $\tanh 2x$ when $\tanh x = \frac{2}{5}$. [5 marks]

12. Find $(\cosh x - \sinh x)^5$ when $x = \ln 5$. [1 mark]

13. If $\tanh x = \frac{3}{7}$, find $\cosh 4x + \sinh 4x$. [3 marks]

14. If $\sinh x = \frac{8}{15}$ and $\cosh y = \frac{25}{24}$, find $\cosh(x + y)\cosh(x - y)$. [2 marks]

15. Prove that $\sinh(x + y)\sinh(x - y) = \sinh^2 x - \sinh^2 y$. Hence find $\sinh(x + y)\sinh(x - y)$ when $\sinh x = \frac{12}{5}$ and $\sinh y = \frac{7}{24}$. [5 marks]

16. Find $\tanh^{-1}\left(\frac{1}{2}\right)$. [2 marks]

17. Find $\sinh^{-1}\left(-\frac{7}{24}\right)$. [3 marks]

18. Prove that $\tanh^{-1} x = \frac{1}{2}\log_e \frac{1+x}{1-x}$, $|x| < 1$. Hence find $\tanh^{-1}\left(\frac{1}{4}\right)$. [5 marks]

19. Find the domain of $f(x) = \coth^{-1}(x - 8)$. [1 mark]

20. Express $\sinh^{-1}\left(\frac{35}{12}\right)$ in terms of $\tanh^{-1}$ and of $\cosh^{-1}$. [3 marks]

Key sheet

Q	Answer
1	$\frac{3}{5}$
2	$\frac{312}{25}$
3	odd
4	$[1, \infty)$
5	$\frac{4}{5}$
6	$-\frac{21}{20}$
7	$\frac{5}{4}$
8	$-\frac{5}{12}$
9	$\frac{97}{72}$
10	$\frac{5}{12}$
11	$\frac{20}{29}$
12	$\frac{1}{3125}$
13	$\frac{25}{4}$
14	$\frac{19721}{14400}$
15	$\frac{81719}{14400}$
16	$\frac{\log_e 3}{2}$
17	$-\log_e \frac{4}{3}$
18	$\frac{\log_e \frac{5}{3}}{2}$
19	$(-\infty, 7) \cup (9, \infty)$
20	$\tanh^{-1}\left(\frac{35}{37}\right) = \cosh^{-1}\left(\frac{37}{12}\right)$

Solutions

1. Find $\tanh(\ln 2)$. [1]

$\tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}}$

$\tanh(\ln 2) = \frac{2 - \frac{1}{2}}{2 + \frac{1}{2}} = \frac{3}{5}$

Ans: $\frac{3}{5}$

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2. Find $2\sinh(\ln 5)\cosh(\ln 5)$. [2]

$\sinh(\ln 5) = \frac{12}{5},\ \cosh(\ln 5) = \frac{13}{5}$

$2 \times \frac{12}{5} \times \frac{13}{5} = \frac{312}{25}$

Ans: $\frac{312}{25}$

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3. The function $f(x) = \sinh^7 x + x\cosh 2x$ is [1]

$\cosh$ and sech are even; $\sinh$ and $\tanh$ are odd.

Replacing x by $-x$: $f(-x) = -f(x)$, so f is odd.

Ans: odd

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4. Find the range of $f(x) = 2\cosh x - 1$. [2]

$\cosh x$ has range $[1, \infty)$.

$2\cosh x - 1 \geq 2(1) - 1 = 1$

Range $= [1, \infty)$

Ans: $[1, \infty)$

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5. If $\cosh x = \frac{5}{4}$, find $\operatorname{sech} x$. [1]

$\operatorname{sech} x = \frac{1}{\cosh x} = \frac{4}{5}$

Ans: $\frac{4}{5}$

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6. If $\cosh x = \frac{29}{20}$ and x is negative, find $\sinh x$. [2]

$\sinh^2 x = \cosh^2 x - 1 = \frac{841}{400} - 1 = \frac{441}{400}$

x is negative, so $\sinh x$ is negative: $\sinh x = -\frac{21}{20}$

Ans: $-\frac{21}{20}$

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7. If $\cosh x = \frac{5}{4}$ and $\sinh y = \frac{15}{8}$, with $x, y > 0$, find $\cosh(x - y)$. [3]

$\sinh x = \sqrt{\frac{25}{16} - 1} = \frac{3}{4},\ \cosh y = \sqrt{1 + \frac{225}{64}} = \frac{17}{8}$

$\cosh(x - y) = \cosh x \cosh y - \sinh x \sinh y$

$= \frac{5}{4} \cdot \frac{17}{8} - \frac{3}{4} \cdot \frac{15}{8} = \frac{5}{4}$

Ans: $\frac{5}{4}$

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8. Prove that $\sinh(x-y) = \sinh x \cosh y - \cosh x \sinh y$. Hence find $\sinh(x-y)$ when $\sinh x = \frac{3}{4}$ and $\cosh y = \frac{5}{3}$ $(x, y > 0)$. [5]

$\text{RHS} = \frac{(e^x - e^{-x})(e^y + e^{-y}) - (e^x + e^{-x})(e^y - e^{-y})}{4}$

$= \frac{(e^{x+y} + e^{x-y} - e^{-x+y} - e^{-x-y}) - (e^{x+y} - e^{x-y} + e^{-x+y} - e^{-x-y})}{4}$

$= \frac{2e^{x-y} - 2e^{-(x-y)}}{4} = \sinh(x-y) = \text{LHS}$

Hence:

$\cosh x = \frac{5}{4},\ \sinh y = \frac{4}{3}$

$\sinh(x-y) = \frac{3}{4} \cdot \frac{5}{3} - \frac{5}{4} \cdot \frac{4}{3} = -\frac{5}{12}$

Ans: $-\frac{5}{12}$

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9. If $\cosh x = \frac{13}{12}$, find $\cosh 2x$. [1]

$\cosh 2x = 2\cosh^2 x - 1 = 2 \cdot \frac{169}{144} - 1 = \frac{97}{72}$

Ans: $\frac{97}{72}$

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10. If $\tanh x = \frac{1}{5}$, find $\sinh 2x$. [2]

$\sinh 2x = \frac{2\tanh x}{1 - \tanh^2 x} = \frac{\frac{2}{5}}{1 - \frac{1}{25}} = \frac{5}{12}$

Ans: $\frac{5}{12}$

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11. Prove that $\tanh 2x = \frac{2\tanh x}{1 + \tanh^2 x}$. Hence find $\tanh 2x$ when $\tanh x = \frac{2}{5}$. [5]

$\tanh 2x = \frac{\sinh 2x}{\cosh 2x} = \frac{2\sinh x \cosh x}{\cosh^2 x + \sinh^2 x}$

Divide the numerator and the denominator by $\cosh^2 x$:

$= \frac{2\tanh x}{1 + \tanh^2 x}$

Hence $\tanh 2x = \frac{20}{29}$

Ans: $\frac{20}{29}$

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12. Find $(\cosh x - \sinh x)^5$ when $x = \ln 5$. [1]

$(\cosh x - \sinh x)^5 = e^{-5x}$

$= e^{-5\ln 5} = 5^{-5} = \frac{1}{3125}$

Ans: $\frac{1}{3125}$

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13. If $\tanh x = \frac{3}{7}$, find $\cosh 4x + \sinh 4x$. [3]

$\cosh 2nx + \sinh 2nx = \left(\frac{1+\tanh x}{1-\tanh x}\right)^n$ with $n = 2$

$\frac{1 + \frac{3}{7}}{1 - \frac{3}{7}} = \frac{5}{2}$

$\text{Answer} = \left(\frac{5}{2}\right)^2 = \frac{25}{4}$

Ans: $\frac{25}{4}$

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14. If $\sinh x = \frac{8}{15}$ and $\cosh y = \frac{25}{24}$, find $\cosh(x+y)\cosh(x-y)$. [2]

$\cosh(x+y)\cosh(x-y) = \sinh^2 x + \cosh^2 y$

$= \frac{64}{225} + \frac{625}{576} = \frac{19721}{14400}$

Ans: $\frac{19721}{14400}$

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15. Prove that $\sinh(x+y)\sinh(x-y) = \sinh^2 x - \sinh^2 y$. Hence find $\sinh(x+y)\sinh(x-y)$ when $\sinh x = \frac{12}{5}$ and $\sinh y = \frac{7}{24}$. [5]

Multiply: $(\sinh x\cosh y + \cosh x\sinh y)(\sinh x\cosh y - \cosh x\sinh y) = \sinh^2 x\cosh^2 y - \cosh^2 x\sinh^2 y$.

Put $\cosh^2 y = 1+\sinh^2 y$ and $\cosh^2 x = 1+\sinh^2 x$:

$= \sinh^2 x(1+\sinh^2 y) - (1+\sinh^2 x)\sinh^2 y = \sinh^2 x - \sinh^2 y$

Hence $\frac{144}{25} - \frac{49}{576} = \frac{81719}{14400}$

Ans: $\frac{81719}{14400}$

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16. Find $\tanh^{-1}\left(\frac{1}{2}\right)$. [2]

$\tanh^{-1} x = \frac{1}{2}\log_e \frac{1+x}{1-x}$

$= \frac{1}{2}\log_e \frac{\frac{3}{2}}{\frac{1}{2}} = \frac{\log_e 3}{2}$

Ans: $\frac{\log_e 3}{2}$

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17. Find $\sinh^{-1}\left(-\frac{7}{24}\right)$. [3]

$\sinh^{-1} x = \log_e\left(x + \sqrt{x^2+1}\right)$

$\sqrt{\frac{49}{576}+1} = \frac{25}{24}$

$\sinh^{-1}\left(-\frac{7}{24}\right) = \log_e\left(-\frac{7}{24}+\frac{25}{24}\right) = -\log_e\frac{4}{3}$

Ans: $-\log_e\frac{4}{3}$

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18. Prove that $\tanh^{-1} x = \frac{1}{2}\log_e\frac{1+x}{1-x}$, $|x|<1$. Hence find $\tanh^{-1}\left(\frac{1}{4}\right)$. [5]

Let $y = \tanh^{-1} x$, so $x = \frac{e^y - e^{-y}}{e^y + e^{-y}} = \frac{e^{2y}-1}{e^{2y}+1}$.

$xe^{2y} + x = e^{2y} - 1 \Rightarrow e^{2y}(1-x) = 1+x$.

$e^{2y} = \frac{1+x}{1-x} \Rightarrow y = \frac{1}{2}\log_e\frac{1+x}{1-x}$.

Hence $\tanh^{-1}\left(\frac{1}{4}\right) = \frac{\log_e\frac{5}{3}}{2}$

Ans: $\frac{\log_e\frac{5}{3}}{2}$

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19. Find the domain of $f(x) = \coth^{-1}(x-8)$. [1]

$\coth^{-1} u$ needs $|u|>1$.

$|x-8|>1 \Rightarrow x<7$ or $x>9$

Ans: $(-\infty, 7)\cup(9,\infty)$

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20. Express $\sinh^{-1}\left(\frac{35}{12}\right)$ in terms of $\tanh^{-1}$ and of $\cosh^{-1}$. [3]

$\sinh^{-1} x = \tanh^{-1}\frac{x}{\sqrt{1+x^2}} = \cosh^{-1}\sqrt{x^2+1}$ for $x>0$.

$\sqrt{1+\frac{1225}{144}} = \frac{37}{12}$

$\tanh^{-1}\frac{\frac{35}{12}}{\frac{37}{12}} = \tanh^{-1}\left(\frac{35}{37}\right)$

$\sinh^{-1}\left(\frac{35}{12}\right) = \tanh^{-1}\left(\frac{35}{37}\right) = \cosh^{-1}\left(\frac{37}{12}\right)$

Ans: $\tanh^{-1}\left(\frac{35}{37}\right) = \cosh^{-1}\left(\frac{37}{12}\right)$
