EXPRESSWALA ACADEMY

Maths 1A: Hyperbolic Functions

Question Paper, Set 01

Time: 1 hour 30 minutes | Maximum marks: 50

Name: _____ Class: _____ Date: _____

1. Answer all 20 questions. The marks for each question are shown in brackets.
2. Show your working for the 2, 3 and 5 mark questions.
3. The key sheet and worked solutions are at the end of this paper.

1. Find $\sinh(-\ln 4)$. [1 mark]

2. Find $\cosh(\ln \frac{4}{3}) + \sinh(\ln 5)$. [2 marks]

3. The function $f(x) = x^4 \cosh 3x + \tanh 3x$ is [1 mark]

4. Find the range of $f(x) = 6 - 5\cosh x$. [2 marks]

5. Simplify $\cosh x + \sinh x$ when $x = \ln \frac{3}{2}$. [1 mark]

6. If $\sinh x = -\frac{8}{15}$, find $\cosh x$ and $\tanh x$. [2 marks]

7. If $\cosh x = \frac{5}{4}$ and $\sinh y = \frac{12}{5}$, with $x, y > 0$, find $\cosh(x+y)$. [3 marks]

8. Prove that $\sinh(x+y) = \sinh x \cosh y + \cosh x \sinh y$. Hence find $\sinh(x+y)$ when $\sinh x = \frac{7}{24}$ and $\cosh y = \frac{5}{3}$ $(x, y > 0)$. [5 marks]

9. If $\sinh x = \frac{3}{4}$ and $\cosh x = \frac{5}{4}$, find $\cosh 2x$. [1 mark]

10. If $\tanh x = \frac{2}{5}$, find $\cosh 2x$. [2 marks]

11. Prove that $\tanh 2x = \frac{2\tanh x}{1+\tanh^2 x}$. Hence find $\tanh 2x$ when $\tanh x = \frac{1}{3}$. [5 marks]

12. Find $(\cosh x + \sinh x)^3$ when $x = \ln 2$. [1 mark]

13. Using the series, write the first three non-zero terms of $\sinh 4x$. [3 marks]

14. If $\sinh x = \frac{21}{20}$ and $\sinh y = \frac{15}{8}$, find $\sinh(x+y)\sinh(x-y)$. [2 marks]

15. Prove that $\sinh(x+y) + \sinh(x-y) = 2\sinh x \cosh y$. Hence find $\sinh(x+y) + \sinh(x-y)$ when $\sinh x = \frac{4}{3}$ and $\cosh y = \frac{5}{4}$ $(x, y > 0)$. [5 marks]

16. Find $\tanh^{-1}\left(\frac{3}{5}\right)$. [2 marks]

17. Find $\sinh^{-1}\left(\frac{35}{12}\right)$. [3 marks]

18. Prove that $\tanh^{-1} x = \frac{1}{2}\log_e \frac{1+x}{1-x}$, $|x| < 1$. Hence find $\tanh^{-1}\left(\frac{3}{4}\right)$. [5 marks]

19. Find the domain of $f(x) = \cosh^{-1}(x-6)$. [1 mark]

20. Express $\cosh^{-1}\left(\frac{17}{8}\right)$ in terms of $\sinh^{-1}$ and of sech^{-1}. [3 marks]

Key sheet

Q	Answer
1	$-\frac{15}{8}$
2	$\frac{413}{120}$
3	neither even nor odd
4	$(-\infty, 1]$
5	$\frac{3}{2}$
6	$\cosh x = \frac{17}{15},\ \tanh x = -\frac{8}{17}$
7	$\frac{101}{20}$
8	$\frac{15}{8}$
9	$\frac{17}{8}$
10	$\frac{29}{21}$
11	$\frac{3}{5}$
12	8
13	$\sinh 4x \approx \frac{128x^5}{15} + \frac{32x^3}{3} + 4x$
14	$-\frac{3861}{1600}$
15	$\frac{10}{3}$
16	$\frac{\log_e 4}{2}$
17	$\log_e 6$
18	$\frac{\log_e 7}{2}$
19	$[7, \infty)$
20	$\sinh^{-1}\left(\frac{15}{8}\right) = \operatorname{sech}^{-1}\left(\frac{8}{17}\right)$

Solutions

1. Find $\sinh(-\ln 4)$. [1]

$\sinh x = \frac{e^x - e^{-x}}{2}$

$e^{\ln 4} = 4,\ e^{-\ln 4} = \frac{1}{4}$

$\sinh(-\ln 4) = -\frac{4 - \frac{1}{4}}{2} = -\frac{15}{8}$

Ans: $-\frac{15}{8}$

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2. Find $\cosh(\ln \frac{4}{3}) + \sinh(\ln 5)$. [2]

$\cosh(\ln \frac{4}{3}) = \frac{\frac{4}{3} + \frac{3}{4}}{2} = \frac{25}{24}$

$\sinh(\ln 5) = \frac{5 - \frac{1}{5}}{2} = \frac{12}{5}$

$\text{Sum} = \frac{413}{120}$

Ans: $\frac{413}{120}$

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3. The function $f(x) = x^4 \cosh 3x + \tanh 3x$ is [1]

$\cosh$ and sech are even; $\sinh$ and $\tanh$ are odd.

Replacing x by $-x$: one term keeps its sign and the other changes, so f is neither.

Ans: neither even nor odd

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4. Find the range of $f(x) = 6 - 5\cosh x$. [2]

$\cosh x \geq 1$, so $-5\cosh x \leq -5$.

$f(x) \leq 6 - 5 = 1$

Range $= (-\infty, 1]$

Ans: $(-\infty, 1]$

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5. Simplify $\cosh x + \sinh x$ when $x = \ln \frac{3}{2}$. [1]

$\cosh x + \sinh x = e^x$

$e^{\ln \frac{3}{2}} = \frac{3}{2}$

Ans: $\frac{3}{2}$

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6. If $\sinh x = -\frac{8}{15}$, find $\cosh x$ and $\tanh x$. [2]

$\cosh^2 x = 1 + \sinh^2 x = 1 + \frac{64}{225} = \frac{289}{225}$

Since $\cosh x \geq 1$, $\cosh x = \frac{17}{15}$

$\tanh x = \frac{\sinh x}{\cosh x} = \frac{-\frac{8}{15}}{\frac{17}{15}} = -\frac{8}{17}$

Ans: $\cosh x = \frac{17}{15},\ \tanh x = -\frac{8}{17}$

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7. If $\cosh x = \frac{5}{4}$ and $\sinh y = \frac{12}{5}$, with $x, y > 0$, find $\cosh(x + y)$. [3]

$\sinh x = \sqrt{\frac{25}{16} - 1} = \frac{3}{4},\ \cosh y = \sqrt{1 + \frac{144}{25}} = \frac{13}{5}$

$\cosh(x + y) = \cosh x \cosh y + \sinh x \sinh y$

$= \frac{5}{4} \cdot \frac{13}{5} + \frac{3}{4} \cdot \frac{12}{5} = \frac{101}{20}$

Ans: $\frac{101}{20}$

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8. Prove that $\sinh(x+y) = \sinh x \cosh y + \cosh x \sinh y$. Hence find $\sinh(x+y)$ when $\sinh x = \frac{7}{24}$ and $\cosh y = \frac{5}{3}$ $(x, y > 0)$. [5]

$\text{RHS} = \frac{(e^x - e^{-x})(e^y + e^{-y}) + (e^x + e^{-x})(e^y - e^{-y})}{4}$

$= \frac{(e^{x+y} + e^{x-y} - e^{-x+y} - e^{-x-y}) + (e^{x+y} - e^{x-y} + e^{-x+y} - e^{-x-y})}{4}$

$= \frac{2e^{x+y} - 2e^{-(x+y)}}{4} = \frac{e^{x+y} - e^{-(x+y)}}{2} = \sinh(x+y) = \text{LHS}$

Hence:

$\cosh x = \frac{25}{24},\ \sinh y = \frac{4}{3}$

$\sinh(x+y) = \frac{7}{24} \cdot \frac{5}{3} + \frac{25}{24} \cdot \frac{4}{3} = \frac{15}{8}$

Ans: $\frac{15}{8}$

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9. If $\sinh x = \frac{3}{4}$ and $\cosh x = \frac{5}{4}$, find $\cosh 2x$. [1]

$\cosh 2x = \cosh^2 x + \sinh^2 x = \frac{25}{16} + \frac{9}{16} = \frac{17}{8}$

Ans: $\frac{17}{8}$

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10. If $\tanh x = \frac{2}{5}$, find $\cosh 2x$. [2]

$\cosh 2x = \frac{1 + \tanh^2 x}{1 - \tanh^2 x} = \frac{1 + \frac{4}{25}}{1 - \frac{4}{25}} = \frac{29}{21}$

Ans: $\frac{29}{21}$

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11. Prove that $\tanh 2x = \frac{2\tanh x}{1 + \tanh^2 x}$. Hence find $\tanh 2x$ when $\tanh x = \frac{1}{3}$. [5]

$\tanh 2x = \frac{\sinh 2x}{\cosh 2x} = \frac{2\sinh x \cosh x}{\cosh^2 x + \sinh^2 x}$

Divide the numerator and the denominator by $\cosh^2 x$:

$= \frac{2\tanh x}{1 + \tanh^2 x}$

Hence $\tanh 2x = \frac{3}{5}$

Ans: $\frac{3}{5}$

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12. Find $(\cosh x + \sinh x)^3$ when $x = \ln 2$. [1]

$(\cosh x + \sinh x)^3 = e^{3x}$

$= e^{3\ln 2} = 2^3 = 8$

Ans: 8

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13. Using the series, write the first three non-zero terms of $\sinh 4x$. [3]

$\sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots$

Replace x by $4x$:

$\sinh 4x = 4x + \frac{(4x)^3}{6} + \frac{(4x)^5}{120} + \cdots$

$= 4x + \frac{32x^3}{3} + \frac{128x^5}{15} + \cdots$

Ans: $\sinh 4x \approx \frac{128x^5}{15} + \frac{32x^3}{3} + 4x$

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14. If $\sinh x = \frac{21}{20}$ and $\sinh y = \frac{15}{8}$, find $\sinh(x+y)\sinh(x-y)$. [2]

$\sinh(x+y)\sinh(x-y) = \sinh^2 x - \sinh^2 y$

$= \frac{441}{400} - \frac{225}{64} = -\frac{3861}{1600}$

Ans: $-\frac{3861}{1600}$

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15. Prove that $\sinh(x+y)+\sinh(x-y)=2\sinh x\cosh y$. Hence find $\sinh(x+y)+\sinh(x-y)$ when $\sinh x=\frac{4}{3}$ and $\cosh y=\frac{5}{4}$ $(x,y>0)$. [5]

Add the formulas for $\sinh(x+y)$ and $\sinh(x-y)$: the $\cosh x\sinh y$ terms cancel.

So the left side equals $2\sinh x\cosh y$.

Hence $2\sinh x\cosh y=2\cdot\frac{4}{3}\cdot\frac{5}{4}=\frac{10}{3}$

Ans: $\frac{10}{3}$

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16. Find $\tanh^{-1}\left(\frac{3}{5}\right)$. [2]

$\tanh^{-1}x=\frac{1}{2}\log_e\frac{1+x}{1-x}$

$=\frac{1}{2}\log_e\frac{\frac{8}{5}}{\frac{2}{5}}=\frac{\log_e 4}{2}$

Ans: $\frac{\log_e 4}{2}$

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17. Find $\sinh^{-1}\left(\frac{35}{12}\right)$. [3]

$\sinh^{-1}x=\log_e\left(x+\sqrt{x^2+1}\right)$

$\sqrt{\frac{1225}{144}+1}=\frac{37}{12}$

$\sinh^{-1}\left(\frac{35}{12}\right)=\log_e\left(\frac{35}{12}+\frac{37}{12}\right)=\log_e 6$

Ans: $\log_e 6$

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18. Prove that $\tanh^{-1}x=\frac{1}{2}\log_e\frac{1+x}{1-x}$, $|x|<1$. Hence find $\tanh^{-1}\left(\frac{3}{4}\right)$. [5]

Let $y=\tanh^{-1}x$, so $x=\frac{e^y-e^{-y}}{e^y+e^{-y}}=\frac{e^{2y}-1}{e^{2y}+1}$.

$xe^{2y}+x=e^{2y}-1\Rightarrow e^{2y}(1-x)=1+x$.

$e^{2y}=\frac{1+x}{1-x}\Rightarrow y=\frac{1}{2}\log_e\frac{1+x}{1-x}$.

Hence $\tanh^{-1}\left(\frac{3}{4}\right)=\frac{\log_e 7}{2}$

Ans: $\frac{\log_e 7}{2}$

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19. Find the domain of $f(x)=\cosh^{-1}(x-6)$. [1]

$\cosh^{-1}u$ needs $u\geq 1$.

$x-6\geq 1\Rightarrow x\geq 7$

Ans: $[7,\infty)$

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20. Express $\cosh^{-1}\left(\frac{17}{8}\right)$ in terms of $\sinh^{-1}$ and of sech^{-1}. [3]

$\cosh^{-1}x=\sinh^{-1}\sqrt{x^2-1}=\operatorname{sech}^{-1}\frac{1}{x}$ for $x\geq 1$.

$\sqrt{\frac{289}{64}-1}=\frac{15}{8}$, $\frac{1}{\frac{17}{8}}=\frac{8}{17}$

$\cosh^{-1}\left(\frac{17}{8}\right)=\sinh^{-1}\left(\frac{15}{8}\right)=\operatorname{sech}^{-1}\left(\frac{8}{17}\right)$

Ans: $\sinh^{-1}\left(\frac{15}{8}\right)=\operatorname{sech}^{-1}\left(\frac{8}{17}\right)$
