EXPRESSWALA ACADEMY

Maths 1A: Functions

Question Paper, Set 09

Time: 1 hour 30 minutes | Maximum marks: 50

Name: _____ Class: _____ Date: _____

1. Answer all 20 questions. The marks for each question are shown in brackets.
2. Show your working for the 2, 3 and 5 mark questions.
3. The key sheet and worked solutions are at the end of this paper.

1. If $A = \{2, 3, 9\}$ and $B = \{t, u\}$, how many functions are there from A to B? [1 mark]

2. If $f : A \to \mathbb{Z}$ is defined by $f(x) = x^2 + 5$ where $A = \{-2, 0, 2, 4\}$, find the range of f. [1 mark]

3. Find the point where the line $y = x - 7$ meets the x-axis. [1 mark]

4. Find the minimum value of $f(x) = 3x^2 - 24x + 42$. [2 marks]

5. Solve $x^2 + 8x + 20 < 0$. [2 marks]

6. Solve $|3x + 1| \geq 1$. [2 marks]

7. Find the value of $[1.9] + [-6.6]$, where $[x]$ is the greatest integer function. [1 mark]

8. Using $[x] + \left[x + \frac{1}{n}\right] + \cdots + \left[x + \frac{n-1}{n}\right] = [nx]$, evaluate $[0.59] + \left[0.59 + \frac{1}{4}\right] + \left[0.59 + \frac{2}{4}\right] + \left[0.59 + \frac{3}{4}\right]$. [3 marks]

9. Find the value of $6^{3\log_6 30}$. [1 mark]

10. If $f(x) = -2x - 5$ and $g(x) = -x^2 - 3$, find $(g \circ f)(x)$. [2 marks]

11. If $f(x) = \dfrac{-x - 2}{2x + 1}$, show that $(f \circ f)(x) = x$ and hence find $(f \circ f \circ f)(2)$. [5 marks]

12. Write $f(x) = x^4 + 5x^3 - 3x^2 + 4x - 1$ as the sum of an even function and an odd function. [3 marks]

13. If $f(x) = x + 1$ and $g(x) = x^2 - 9$, find $(fg)(4)$ and $\left(\dfrac{f}{g}\right)(-1)$. [2 marks]

14. A polynomial f satisfies $f(x)f\left(\frac{1}{x}\right) = f(x) + f\left(\frac{1}{x}\right)$ and $f(5) = 26$. Find $f(4)$. [3 marks]

15. How many one-one functions are there from $A = \{1, 4, 10, 12\}$ to $B = \{a, c, e, f, h\}$? [2 marks]

16. Let $A = \mathbb{R} - \{-1\}$ and $B = \mathbb{R} - \left\{-\frac{3}{4}\right\}$. Show that $f : A \to B$, $f(x) = \frac{6 - 3x}{4x + 4}$ is a bijection and find f^{-1}. [5 marks]

17. Find the domain of $f(x) = \sqrt{x^2 - x - 12} + \dfrac{1}{\sqrt{-x - 6}}$. [5 marks]

18. Find the range of $f(x) = \dfrac{x^2 + 2x + 4}{x^2 + 3x + 6}$, $x \in \mathbb{R}$. [5 marks]

19. The point $(-5, 3)$ lies on $y = f(x)$. Which point must lie on $y = f^{-1}(x)$? [1 mark]

20. Where does the graph of $y = |x^2 + 5x + 4|$ meet the x-axis, and what is the highest point of the arch between these points? [3 marks]

Key sheet

Q	Answer
1	8
2	$\text{Range} = \{5, 9, 21\}$
3	$(7, 0)$
4	-6
5	ϕ
6	$x \in (-\infty, -\frac{2}{3}] \cup [0, \infty)$
7	-6
8	2
9	27000
10	$(g \circ f)(x) = -4x^2 - 20x - 28$
11	$(f \circ f)(x) = x$; $(f \circ f \circ f)(2) = -\frac{4}{5}$
12	Even part $x^4 - 3x^2 - 1$, odd part $5x^3 + 4x$
13	$(fg)(4) = 35$, $\left(\frac{f}{g}\right)(-1) = 0$
14	$f(4) = 17$
15	120
16	$f^{-1}(x) = \frac{6-4x}{4x+3}$
17	$(-\infty, -6)$
18	$[\frac{2}{3}, \frac{6}{5}]$
19	$(3, -5)$
20	At $x = -4$ and $x = -1$; the top of the arch is $(-\frac{5}{2}, \frac{9}{4})$

Solutions

1. If $A = \{2, 3, 9\}$ and $B = \{t, u\}$, how many functions are there from A to B? [1]

$n(A) = 3,\ n(B) = 2$

$\text{Number of functions} = \{n(B)\}^{n(A)} = 2^3 = 8$

Ans: 8

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2. If $f : A \to \mathbb{Z}$ is defined by $f(x) = x^2 + 5$ where $A = \{-2, 0, 2, 4\}$, find the range of f. [1]

$f(-2) = 9,\ f(0) = 5,\ f(2) = 9,\ f(4) = 21$

Range $= \{5, 9, 21\}$

Ans: $\text{Range} = \{5, 9, 21\}$

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3. Find the point where the line $y = x - 7$ meets the x-axis. [1]

$y = 0 \Rightarrow x - 7 = 0$

$\Rightarrow x = 7$

The point is $(7, 0)$.

Ans: $(7, 0)$

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4. Find the minimum value of $f(x) = 3x^2 - 24x + 42$. [2]

$\frac{4ac - b^2}{4a} = \frac{4(3)(42) - (-24)^2}{4(3)} = -6$

It occurs at $x = -\frac{b}{2a} = 4$.

Ans: -6

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5. Solve $x^2 + 8x + 20 < 0$. [2]

$\Delta = -16 < 0$

So the expression has the sign of $a = 1$ for every real x.

Solution set: ϕ

Ans: ϕ

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6. Solve $|3x + 1| \geq 1$. [2]

$3x + 1 \geq 1$ or $3x + 1 \leq -1$

Solution set: $(-\infty, -\frac{2}{3}] \cup [0, \infty)$

Ans: $x \in (-\infty, -\frac{2}{3}] \cup [0, \infty)$

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7. Find the value of $[1.9] + [-6.6]$, where $[x]$ is the greatest integer function. [1]

$[1.9] = 1,\ [-6.6] = -7$

$[1.9] + [-6.6] = -6$

Ans: -6

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8. Using $[x] + \left[x + \frac{1}{n}\right] + \cdots + \left[x + \frac{n-1}{n}\right] = [nx]$, evaluate $[0.59] + \left[0.59 + \frac{1}{4}\right] + \left[0.59 + \frac{2}{4}\right] + \left[0.59 + \frac{3}{4}\right]$. [3]

Here $x = 0.59$, $n = 4$.

$\text{Sum} = [4 \times 0.59] = [2.36] = 2$

Check: the terms are 0, 0, 1, 1.

Ans: 2

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9. Find the value of $6^{3\log_6 30}$. [1]

$6^{3\log_6 30} = 6^{\log_6 30^3}$

$= 30^3 = 27000$ (using $a^{\log_a N} = N$)

Ans: 27000

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10. If $f(x) = -2x - 5$ and $g(x) = -x^2 - 3$, find $(g \circ f)(x)$. [2]

$(g \circ f)(x) = g(f(x)) = g(-2x - 5)$

$= -(-2x - 5)^2 - 3$

$= -4x^2 - 20x - 28$

Ans: $(g \circ f)(x) = -4x^2 - 20x - 28$

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11. If $f(x) = \dfrac{-x - 2}{2x + 1}$, show that $(f \circ f)(x) = x$ and hence find $(f \circ f \circ f)(2)$. [5]

$(f \circ f)(x) = f\left(\frac{-x-2}{2x+1}\right) = \dfrac{-1\left(\frac{-x-2}{2x+1}\right) - 2}{2\left(\frac{-x-2}{2x+1}\right) + 1}$

Multiply the numerator and the denominator by $2x + 1$:

$= \dfrac{(-4x - 2) + (x + 2)}{(-2x - 4) + (2x + 1)} = \dfrac{-3x}{-3}$

$= x$

So $f \circ f = I$, and $(f \circ f \circ f)(x) = f(x)$.

$(f \circ f \circ f)(2) = f(2) = \frac{-4}{5} = -\frac{4}{5}$

Ans: $(f \circ f)(x) = x$; $(f \circ f \circ f)(2) = -\frac{4}{5}$

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12. Write $f(x) = x^4 + 5x^3 - 3x^2 + 4x - 1$ as the sum of an even function and an odd function. [3]

$f(x) = \frac{1}{2}[f(x) + f(-x)] + \frac{1}{2}[f(x) - f(-x)]$

$f(-x) = x^4 - 5x^3 - 3x^2 - 4x - 1$

$\frac{1}{2}[f(x) + f(-x)] = x^4 - 3x^2 - 1$ (even)

$\frac{1}{2}[f(x) - f(-x)] = 5x^3 + 4x$ (odd)

$f(x) = \left(x^4 - 3x^2 - 1\right) + \left(5x^3 + 4x\right)$

Ans: Even part $x^4 - 3x^2 - 1$, odd part $5x^3 + 4x$

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13. If $f(x) = x + 1$ and $g(x) = x^2 - 9$, find $(fg)(4)$ and $\left(\dfrac{f}{g}\right)(-1)$. [2]

$(fg)(4) = f(4)\, g(4) = (5)(7) = 35$

$\left(\frac{f}{g}\right)(-1) = \frac{f(-1)}{g(-1)} = \frac{0}{-8} = 0$

Ans: $(fg)(4) = 35$, $\left(\frac{f}{g}\right)(-1) = 0$

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14. A polynomial f satisfies $f(x)f\left(\frac{1}{x}\right) = f(x) + f\left(\frac{1}{x}\right)$ and $f(5) = 26$. Find $f(4)$. [3]

So $f(x) = 1 \pm x^n$.

$1 + 5^n = 26 \Rightarrow 5^n = 25 \Rightarrow n = 2$

$f(x) = 1 + x^2$

$f(4) = 1 + (4)^2 = 17$

Ans: $f(4) = 17$

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15. How many one-one functions are there from $A = \{1, 4, 10, 12\}$ to $B = \{a, c, e, f, h\}$? [2]

$n(A) = 4 \leq n(B) = 5$

$\text{Number} = {}^5P_4 = 120$

Ans: 120

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16. Let $A = \mathbb{R} - \{-1\}$ and $B = \mathbb{R} - \left\{-\frac{3}{4}\right\}$. Show that $f : A \to B$, $f(x) = \frac{6-3x}{4x+4}$ is a bijection and find f^{-1}. [5]

One-one: $f(x_1) = f(x_2) \Rightarrow (6 - 3x_1)(4x_2 + 4) = (6 - 3x_2)(4x_1 + 4)$

$\Rightarrow -36(x_1 - x_2) = 0 \Rightarrow x_1 = x_2$, so f is one-one.

Onto: for $y \in B$, $y = \frac{6-3x}{4x+4} \Rightarrow y(4x + 4) = 6 - 3x$

$\Rightarrow x = \frac{6-4y}{4y+3}$, defined since $y \neq -\frac{3}{4}$, and $x \neq -1$ (else $-12 = 24$).

So $x \in A$ and $f(x) = y$: f is onto. Hence f is a bijection.

$f^{-1}(x) = \frac{6-4x}{4x+3}$

Ans: $f^{-1}(x) = \frac{6-4x}{4x+3}$

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17. Find the domain of $f(x) = \sqrt{x^2 - x - 12} + \dfrac{1}{\sqrt{-x - 6}}$. [5]

We need $x^2 - x - 12 \geq 0$ and $-x - 6 > 0$.

$x^2 - x - 12 = (x - 4)(x + 3) \geq 0$: the roots are $-3 < 4$ and $a > 0$.

$\Rightarrow x \in (-\infty, -3] \cup [4, \infty)$

$-x - 6 > 0 \Rightarrow x < -6$

Common part: domain $= (-\infty, -6)$

Ans: $(-\infty, -6)$

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18. Find the range of $f(x) = \dfrac{x^2 + 2x + 4}{x^2 + 3x + 6}$, $x \in \mathbb{R}$. [5]

The denominator $x^2 + 3x + 6$ has $\Delta = -15 < 0$, so it is never zero: the domain is $\mathbb{R}$.

$y = \frac{x^2+2x+4}{x^2+3x+6} \Rightarrow y(x^2 + 3x + 6) = x^2 + 2x + 4$

$\Rightarrow (y - 1)x^2 + (3y - 2)x + (6y - 4) = 0$

For $y \neq 1$, x is real, so $\Delta = (3y - 2)^2 - 4(y - 1)(6y - 4) \geq 0$

$\Rightarrow -15y^2 + 28y - 12 \geq 0, \Rightarrow \frac{2}{3} \leq y \leq \frac{6}{5}$

The value $y = 1$ is taken at $x = -2$, and 1 lies in this interval.

Range $= [\frac{2}{3}, \frac{6}{5}]$

Ans: $[\frac{2}{3}, \frac{6}{5}]$

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19. The point $(-5, 3)$ lies on $y = f(x)$. Which point must lie on $y = f^{-1}(x)$? [1]

The graph of $y = f^{-1}(x)$ is obtained by reflection in $y = x$: $(x, y) \to (y, x)$.

So $(-5, 3) \to (3, -5)$.

Ans: $(3, -5)$

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20. Where does the graph of $y = |x^2 + 5x + 4|$ meet the x-axis, and what is the highest point of the arch between these points? [3]

$y = x^2 + 5x + 4$ cuts the x-axis where $(x + 1)(x + 4) = 0$: $x = -4, -1$.

Between them it dips to $-\frac{9}{4}$ at $x = -\frac{5}{2}$.

Taking $|f(x)|$ flips this part up: the arch has top $(-\frac{5}{2}, \frac{9}{4})$.

Ans: At $x = -4$ and $x = -1$; the top of the arch is $(-\frac{5}{2}, \frac{9}{4})$
