EXPRESSWALA ACADEMY

Maths 1A: Functions

Question Paper, Set 08

Time: 1 hour 30 minutes | Maximum marks: 50

Name: _____ Class: _____ Date: _____

1. Answer all 20 questions. The marks for each question are shown in brackets.
2. Show your working for the 2, 3 and 5 mark questions.
3. The key sheet and worked solutions are at the end of this paper.

1. If $A = \{1, 3, 6, 8\}$ and $B = \{q, r, t, v, w\}$, how many elements does $A \times B$ have? [1 mark]

2. If $f : A \to \mathbb{Z}$ is defined by $f(x) = x^2 - 2$ where $A = \{-2, 0, 3, 4\}$, find the range of f. [1 mark]

3. Find the degree of the polynomial function $f(x) = 4x^2 + 7x^3 - 4x^6 - 6x^7$. [1 mark]

4. Find the range of $f(x) = -2x^2 - 16x - 25$, $x \in \mathbb{R}$. [2 marks]

5. Solve $3x^2 - 8x - 16 \geq 0$. [2 marks]

6. Solve $|x + 4| > 6$. [2 marks]

7. Find $[2.7] + [-2.7]$. [1 mark]

8. Solve $3[x] + 2 = [x] - 2$, where $[x]$ is the greatest integer function. [3 marks]

9. Find the value of $3^{3\log_3 34}$. [1 mark]

10. If $f(x) = 3x - 4$, find $(f \circ f)(x)$. [2 marks]

11. If $f(x) = \dfrac{5x - 9}{x - 5}$, show that $(f \circ f)(x) = x$ and hence find $(f \circ f \circ f)(-5)$. [5 marks]

12. Write $f(x) = -3x^4 + 3x^3 - 6x^2 + 2x - 5$ as the sum of an even function and an odd function. [3 marks]

13. If $f(x) = x - 1$ and $g(x) = x^2 - 6x + 8$, find the domain of $\dfrac{f}{g}$. [2 marks]

14. f satisfies $f(x + y) = f(x)\, f(y)$ for all real x, y, and $f(1) = \frac{2}{3}$. Find $f(2)$. [3 marks]

15. How many onto functions are there from $A = \{2, 4, 5, 7, 12\}$ to $B = \{d, g\}$? [2 marks]

16. Show that $f : \mathbb{R} \to \mathbb{R}$, $f(x) = \frac{4x+5}{2}$ is a bijection and find f^{-1}. [5 marks]

17. Find the domain of $f(x) = \sqrt{x^2 - 9x + 14} + \dfrac{1}{x - 6}$. [5 marks]

18. Find the range of $f(x) = \dfrac{x^2 + 3x + 4}{x^2 + 2x + 4}$, $x \in \mathbb{R}$. [5 marks]

19. The point $(-8, 7)$ lies on $y = f(x)$. Which point must lie on $y = f(x) + 4$? [1 mark]

20. Where does the graph of $y = |x^2 - 4x - 5|$ meet the x-axis, and what is the highest point of the arch between these points? [3 marks]

Key sheet

Q	Answer
1	20
2	$\text{Range} = \{-2, 2, 7, 14\}$
3	$\text{Degree} = 7$
4	$(-\infty, 7]$
5	$x \in (-\infty, -\frac{4}{3}] \cup [4, \infty)$
6	$x \in (-\infty, -10) \cup (2, \infty)$
7	-1
8	$x \in [-2, -1)$
9	39304
10	$(f \circ f)(x) = 9x - 16$
11	$(f \circ f)(x) = x$; $(f \circ f \circ f)(-5) = \frac{17}{5}$
12	Even part $-3x^4 - 6x^2 - 5$, odd part $3x^3 + 2x$
13	$\mathbb{R} - \{2, 4\}$
14	$f(2) = \frac{4}{9}$
15	30
16	$f^{-1}(x) = \frac{x}{2} - \frac{5}{4}$
17	$(-\infty, 2] \cup [7, \infty)$
18	$[\frac{1}{2}, \frac{7}{6}]$
19	$(-8, 11)$
20	At $x = -1$ and $x = 5$; the top of the arch is $(2, 9)$

Solutions

1. If $A = \{1, 3, 6, 8\}$ and $B = \{q, r, t, v, w\}$, how many elements does $A \times B$ have? [1]

$n(A) = 4,\ n(B) = 5$

$n(A \times B) = n(A) \cdot n(B) = 20$

Ans: 20

---

2. If $f : A \to \mathbb{Z}$ is defined by $f(x) = x^2 - 2$ where $A = \{-2, 0, 3, 4\}$, find the range of f. [1]

$f(-2) = 2,\ f(0) = -2,\ f(3) = 7,\ f(4) = 14$

Range $= \{-2, 2, 7, 14\}$

Ans: $\text{Range} = \{-2, 2, 7, 14\}$

---

3. Find the degree of the polynomial function $f(x) = 4x^2 + 7x^3 - 4x^6 - 6x^7$. [1]

The degree is the highest power of x with a non-zero coefficient.

The highest power is x^7, so the degree is 7.

Ans: $\text{Degree} = 7$

---

4. Find the range of $f(x) = -2x^2 - 16x - 25,\ x \in \mathbb{R}$. [2]

$a = -2 < 0$, so the parabola opens downward.

$f(x) = -2(x + 4)^2 + 7$

The maximum value is 7 at $x = -4$.

Range $= (-\infty, 7]$

Ans: $(-\infty, 7]$

---

5. Solve $3x^2 - 8x - 16 \geq 0$. [2]

$3x^2 - 8x - 16 = (x - 4)(3x + 4)$

The roots are $-\frac{4}{3}$ and 4, and $a = 3 > 0$.

The expression has the sign of a outside the roots.

Solution set: $(-\infty, -\frac{4}{3}] \cup [4, \infty)$

Ans: $x \in (-\infty, -\frac{4}{3}] \cup [4, \infty)$

---

6. Solve $|x + 4| > 6$. [2]

$x + 4 > 6$ or $x + 4 < -6$

Solution set: $(-\infty, -10) \cup (2, \infty)$

Ans: $x \in (-\infty, -10) \cup (2, \infty)$

---

7. Find $[2.7] + [-2.7]$. [1]

$[2.7] = 2,\ [-2.7] = -3$

The sum is -1 (for a non-integer x, $[x] + [-x] = -1$).

Ans: -1

---

8. Solve $3[x] + 2 = [x] - 2$, where $[x]$ is the greatest integer function. [3]

$2[x] = -4 \Rightarrow [x] = -2$

$[x] = -2 \iff -2 \le x < -1$

So $x \in [-2, -1)$.

Ans: $x \in [-2, -1)$

---

9. Find the value of $3^{3\log_3 34}$. [1]

$3^{3\log_3 34} = 3^{\log_3 34^3}$

$= 34^3 = 39304$ (using $a^{\log_a N} = N$)

Ans: 39304

---

10. If $f(x) = 3x - 4$, find $(f \circ f)(x)$. [2]

$(f \circ f)(x) = f(3x - 4) = 3(3x - 4) - 4$

$= 9x - 16$

Ans: $(f \circ f)(x) = 9x - 16$

---

11. If $f(x) = \dfrac{5x - 9}{x - 5}$, show that $(f \circ f)(x) = x$ and hence find $(f \circ f \circ f)(-5)$. [5]

$(f \circ f)(x) = f\left(\frac{5x-9}{x-5}\right) = \dfrac{5\left(\frac{5x-9}{x-5}\right) - 9}{1\left(\frac{5x-9}{x-5}\right) - 5}$

Multiply the numerator and the denominator by $x - 5$:

$= \dfrac{(45 - 9x) + (25x - 45)}{(25 - 5x) + (5x - 9)} = \dfrac{16x}{16}$

$= x$

So $f \circ f = I$, and $(f \circ f \circ f)(x) = f(x)$.

$(f \circ f \circ f)(-5) = f(-5) = \frac{-34}{-10} = \frac{17}{5}$

Ans: $(f \circ f)(x) = x$; $(f \circ f \circ f)(-5) = \frac{17}{5}$

---

12. Write $f(x) = -3x^4 + 3x^3 - 6x^2 + 2x - 5$ as the sum of an even function and an odd function. [3]

$f(x) = \frac{1}{2}[f(x) + f(-x)] + \frac{1}{2}[f(x) - f(-x)]$

$f(-x) = -3x^4 - 3x^3 - 6x^2 - 2x - 5$

$\frac{1}{2}[f(x) + f(-x)] = -3x^4 - 6x^2 - 5$ (even)

$\frac{1}{2}[f(x) - f(-x)] = 3x^3 + 2x$ (odd)

$f(x) = \left(-3x^4 - 6x^2 - 5\right) + \left(3x^3 + 2x\right)$

Ans: Even part $-3x^4 - 6x^2 - 5$**, odd part** $3x^3 + 2x$

---

13. If $f(x) = x - 1$ and $g(x) = x^2 - 6x + 8$, find the domain of $\dfrac{f}{g}$. [2]

$D_f = D_g = \mathbb{R}$, and $\frac{f}{g}$ needs $g(x) \neq 0$.

$(x - 4)(x - 2) = 0 \Rightarrow x = 2, 4$

Domain $= \mathbb{R} - \{2, 4\}$

Ans: $\mathbb{R} - \{2, 4\}$

---

14. f satisfies $f(x+y) = f(x)\,f(y)$ for all real x, y, and $f(1) = \frac{2}{3}$. Find $f(2)$. [3]

$f(2) = f(1+1+\cdots+1) = f(1)^2$

$= \left(\frac{2}{3}\right)^2 = \frac{4}{9}$

Ans: $f(2) = \frac{4}{9}$

---

15. How many onto functions are there from $A = \{2, 4, 5, 7, 12\}$ to $B = \{d, g\}$? [2]

$n(A) = 5,\ n(B) = 2$

$\text{Number of onto functions} = 2^n - 2 = 2^5 - 2 = 30$

Ans: 30

---

16. Show that $f : \mathbb{R} \to \mathbb{R}$, $f(x) = \frac{4x+5}{2}$ is a bijection and find f^{-1}. [5]

One-one: $f(x_1) = f(x_2) \Rightarrow \frac{4x_1+5}{2} = \frac{4x_2+5}{2}$

$\Rightarrow 4x_1 = 4x_2 \Rightarrow x_1 = x_2$, so f is one-one.

Onto: let $y \in \mathbb{R}$ and $y = \frac{4x+5}{2}$.

$\Rightarrow x = \frac{y}{2} - \frac{5}{4} \in \mathbb{R}$

and $f(x) = y$, so every y has a pre-image: f is onto.

Hence f is a bijection, and $f^{-1}(x) = \frac{x}{2} - \frac{5}{4}$

Ans: $f^{-1}(x) = \frac{x}{2} - \frac{5}{4}$

---

17. Find the domain of $f(x) = \sqrt{x^2 - 9x + 14} + \dfrac{1}{x-6}$. [5]

We need $x^2 - 9x + 14 \geq 0$ and $x - 6 \neq 0$.

$x^2 - 9x + 14 = (x-7)(x-2) \geq 0$: the roots are $2 < 7$ and $a > 0$.

$\Rightarrow x \in (-\infty, 2] \cup [7, \infty)$

$x - 6 \neq 0 \Rightarrow x \neq 6$

Common part: domain $= (-\infty, 2] \cup [7, \infty)$

Ans: $(-\infty, 2] \cup [7, \infty)$

---

18. Find the range of $f(x) = \dfrac{x^2 + 3x + 4}{x^2 + 2x + 4}$, $x \in \mathbb{R}$. [5]

The denominator $x^2 + 2x + 4$ has $\Delta = -12 < 0$, so it is never zero: the domain is $\mathbb{R}$.

$y = \frac{x^2+3x+4}{x^2+2x+4} \Rightarrow y(x^2 + 2x + 4) = x^2 + 3x + 4$

$\Rightarrow (y-1)x^2 + (2y-3)x + (4y-4) = 0$

For $y \neq 1$, x is real, so $\Delta = (2y-3)^2 - 4(y-1)(4y-4) \geq 0$

$\Rightarrow -12y^2 + 20y - 7 \geq 0$, $\Rightarrow \frac{1}{2} \leq y \leq \frac{7}{6}$

The value $y = 1$ is taken at $x = 0$, and 1 lies in this interval.

Range $= [\frac{1}{2}, \frac{7}{6}]$

Ans: $[\frac{1}{2}, \frac{7}{6}]$

---

19. The point $(-8, 7)$ lies on $y = f(x)$. Which point must lie on $y = f(x) + 4$? [1]

The graph of $y = f(x) + 4$ is obtained by shift up by 4: $(x, y) \to (x, y+4)$.

So $(-8, 7) \to (-8, 11)$.

Ans: $(-8, 11)$

---

20. Where does the graph of $y = |x^2 - 4x - 5|$ meet the x-axis, and what is the highest point of the arch between these points? [3]

$y = x^2 - 4x - 5$ cuts the x-axis where $(x - 5)(x + 1) = 0$: $x = -1, 5$.

Between them it dips to -9 at $x = 2$.

Taking $|f(x)|$ flips this part up: the arch has top $(2, 9)$.

Ans: At $x = -1$ and $x = 5$; the top of the arch is $(2, 9)$
