EXPRESSWALA ACADEMY

Maths 1A: Functions

Question Paper, Set 07

Time: 1 hour 30 minutes | Maximum marks: 50

Name: _____ Class: _____ Date: _____

1. Answer all 20 questions. The marks for each question are shown in brackets.
2. Show your working for the 2, 3 and 5 mark questions.
3. The key sheet and worked solutions are at the end of this paper.

1. If $A = \{4, 7, 8, 9\}$ and $B = \{q, s, u, w\}$, how many elements does $A \times B$ have? [1 mark]

2. If $f : A \to \mathbb{Z}$ is defined by $f(x) = x^2 - 1$ where $A = \{-3, 0, 1, 2, 4\}$, find the range of f. [1 mark]

3. Find the degree of the polynomial function $f(x) = -6x^4 + 9x^5 + 8x - 4x^6$. [1 mark]

4. Find the range of $f(x) = -x^2 - 6x - 13$, $x \in \mathbb{R}$. [2 marks]

5. Solve $x^2 - 2x + 5 > 0$. [2 marks]

6. Solve $|4x + 4| \geq 1$. [2 marks]

7. Find the value of $[-2.4] + [-7.9]$, where $[x]$ is the greatest integer function. [1 mark]

8. Solve $[x]^2 - [x] - 20 = 0$, where $[x]$ is the greatest integer function. [3 marks]

9. Find $\operatorname{sgn}(x^2 - 3x)$ at $x = \frac{25}{2}$. [1 mark]

10. If $f(x) = 5x + 2$ and $g(x) = 2 - x^2$, find $(g \circ f)(x)$. [2 marks]

11. If $f(x) = \dfrac{-4x - 5}{x + 4}$, show that $(f \circ f)(x) = x$ and hence find $(f \circ f \circ f)(4)$. [5 marks]

12. Write $f(x) = -2x^4 - 6x^3 + x^2 - x - 2$ as the sum of an even function and an odd function. [3 marks]

13. If $f(x) = x + 6$ and $g(x) = x^2 - x - 12$, find the domain of $\dfrac{f}{g}$. [2 marks]

14. A polynomial f satisfies $f(x)f\left(\frac{1}{x}\right) = f(x) + f\left(\frac{1}{x}\right)$ and $f(2) = 17$. Find $f(5)$. [3 marks]

15. How many one-one functions are there from $A = \{2, 5, 10, 11\}$ to $B = \{a, c, d, f, h\}$? [2 marks]

16. Let $A = \mathbb{R} - \{3\}$ and $B = \mathbb{R} - \{-1\}$. Show that $f : A \to B$, $f(x) = \frac{-x-7}{x-3}$ is a bijection and find f^{-1}. [5 marks]

17. Find the domain of $f(x) = \sqrt{x^2 + 9x + 20} + \dfrac{1}{\sqrt{-x - 2}}$. [5 marks]

18. Find the range of $f(x) = \dfrac{x^2 - 3x + 3}{x^2 - 2x + 2}$, $x \in \mathbb{R}$. [5 marks]

19. The point $(9, -1)$ lies on $y = f(x)$. Which point must lie on $y = f(x - 4)$? [1 mark]

20. Describe how to get the graph of $y = \sqrt{x + 1} - 1$ from $y = \sqrt{x}$, and find its domain and range. [3 marks]

Key sheet

Q	Answer
1	16
2	$\text{Range} = \{-1, 0, 3, 8, 15\}$
3	$\text{Degree} = 6$
4	$(-\infty, -4]$
5	$x \in \mathbb{R}$
6	$x \in (-\infty, -\frac{5}{4}] \cup [-\frac{3}{4}, \infty)$
7	-11
8	$x \in [-4, -3) \cup [5, 6)$
9	1
10	$(g \circ f)(x) = -25x^2 - 20x - 2$
11	$(f \circ f)(x) = x$; $(f \circ f \circ f)(4) = -\frac{21}{8}$
12	Even part $-2x^4 + x^2 - 2$, odd part $-6x^3 - x$
13	$\mathbb{R} - \{-3, 4\}$
14	$f(5) = 626$
15	120
16	$f^{-1}(x) = \frac{3x-7}{x+1}$
17	$(-\infty, -5] \cup [-4, -2)$
18	$[\frac{1}{2}, \frac{3}{2}]$
19	$(13, -1)$
20	Domain $[-1, \infty)$, range $[-1, \infty)$

Solutions

1. If $A = \{4, 7, 8, 9\}$ and $B = \{q, s, u, w\}$, how many elements does $A \times B$ have? [1]

$n(A) = 4,\ n(B) = 4$

$n(A \times B) = n(A) \cdot n(B) = 16$

Ans: 16

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2. If $f : A \to \mathbb{Z}$ is defined by $f(x) = x^2 - 1$ where $A = \{-3, 0, 1, 2, 4\}$, find the range of f. [1]

$f(-3) = 8,\ f(0) = -1,\ f(1) = 0,\ f(2) = 3,\ f(4) = 15$

Range $= \{-1, 0, 3, 8, 15\}$

Ans: $\text{Range} = \{-1, 0, 3, 8, 15\}$

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3. Find the degree of the polynomial function $f(x) = -6x^4 + 9x^5 + 8x - 4x^6$. [1]

The degree is the highest power of x with a non-zero coefficient.

The highest power is x^6, so the degree is 6.

Ans: $\text{Degree} = 6$

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4. Find the range of $f(x) = -x^2 - 6x - 13$, $x \in \mathbb{R}$. [2]

$a = -1 < 0$, so the parabola opens downward.

$f(x) = -(x + 3)^2 - 4$

The maximum value is -4 at $x = -3$.

Range $= (-\infty, -4]$

Ans: $(-\infty, -4]$

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5. Solve $x^2 - 2x + 5 > 0$. [2]

$\Delta = -16 < 0$

So the expression has the sign of $a = 1$ for every real x.

Solution set: $\mathbb{R}$

Ans: $x \in \mathbb{R}$

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6. Solve $|4x + 4| \geq 1$. [2]

$4x + 4 \geq 1$ or $4x + 4 \leq -1$

Solution set: $(-\infty, -\frac{5}{4}] \cup [-\frac{3}{4}, \infty)$

Ans: $x \in (-\infty, -\frac{5}{4}] \cup [-\frac{3}{4}, \infty)$

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7. Find the value of $[-2.4] + [-7.9]$, where $[x]$ is the greatest integer function. [1]

$[-2.4] = -3,\ [-7.9] = -8$

$[-2.4] + [-7.9] = -11$

Ans: -11

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8. Solve $[x]^2 - [x] - 20 = 0$, where $[x]$ is the greatest integer function. [3]

Put $t = [x]$: $t^2 - t - 20 = 0$

$(t+4)(t-5) = 0 \Rightarrow t = -4$ or 5

$[x] = -4 \Rightarrow x \in [-4, -3)$, $[x] = 5 \Rightarrow x \in [5, 6)$

So $x \in [-4, -3) \cup [5, 6)$.

Ans: $x \in [-4, -3) \cup [5, 6)$

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9. Find $\text{sgn}(x^2 - 3x)$ at $x = \frac{25}{2}$. [1]

$x^2 - 3x = \frac{475}{4}$ at $x = \frac{25}{2}$

This is positive, so sgn is 1.

Ans: 1

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10. If $f(x) = 5x + 2$ and $g(x) = 2 - x^2$, find $(g \circ f)(x)$. [2]

$(g \circ f)(x) = g(f(x)) = g(5x+2)$

$= -(5x+2)^2 + 2$

$= -25x^2 - 20x - 2$

Ans: $(g \circ f)(x) = -25x^2 - 20x - 2$

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11. If $f(x) = \dfrac{-4x-5}{x+4}$, show that $(f \circ f)(x) = x$ and hence find $(f \circ f \circ f)(4)$. [5]

$(f \circ f)(x) = f\left(\frac{-4x-5}{x+4}\right) = \dfrac{-4\left(\frac{-4x-5}{x+4}\right) - 5}{1\left(\frac{-4x-5}{x+4}\right) + 4}$

Multiply the numerator and the denominator by $x+4$:

$= \dfrac{(-5x-20) + (16x+20)}{(-4x-5) + (4x+16)} = \dfrac{11x}{11}$

$= x$

So $f \circ f = I$, and $(f \circ f \circ f)(x) = f(x)$.

$(f \circ f \circ f)(4) = f(4) = \frac{-21}{8} = -\frac{21}{8}$

Ans: $(f \circ f)(x) = x$; $(f \circ f \circ f)(4) = -\frac{21}{8}$

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12. Write $f(x) = -2x^4 - 6x^3 + x^2 - x - 2$ as the sum of an even function and an odd function. [3]

$f(x) = \frac{1}{2}[f(x) + f(-x)] + \frac{1}{2}[f(x) - f(-x)]$

$f(-x) = -2x^4 + 6x^3 + x^2 + x - 2$

$\frac{1}{2}[f(x) + f(-x)] = -2x^4 + x^2 - 2$ (even)

$\frac{1}{2}[f(x) - f(-x)] = -6x^3 - x$ (odd)

$f(x) = \left(-2x^4 + x^2 - 2\right) + \left(-6x^3 - x\right)$

Ans: Even part $-2x^4 + x^2 - 2$, odd part $-6x^3 - x$

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13. If $f(x) = x + 6$ and $g(x) = x^2 - x - 12$, find the domain of $\dfrac{f}{g}$. [2]

$D_f = D_g = \mathbb{R}$, and $\frac{f}{g}$ needs $g(x) \neq 0$.

$(x-4)(x+3) = 0 \Rightarrow x = -3, 4$

Domain $= \mathbb{R} - \{-3, 4\}$

Ans: $\mathbb{R} - \{-3, 4\}$

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14. A polynomial f satisfies $f(x)f\left(\frac{1}{x}\right) = f(x) + f\left(\frac{1}{x}\right)$ and $f(2) = 17$. Find $f(5)$. [3]

So $f(x) = 1 \pm x^n$.

$1 + 2^n = 17 \Rightarrow 2^n = 16 \Rightarrow n = 4$

$f(x) = 1 + x^4$

$f(5) = 1 + (5)^4 = 626$

Ans: $f(5) = 626$

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15. How many one-one functions are there from $A = \{2, 5, 10, 11\}$ to $B = \{a, c, d, f, h\}$? [2]

$n(A) = 4 \leq n(B) = 5$

$\text{Number} = {}^5P_4 = 120$

Ans: 120

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16. Let $A = \mathbb{R} - \{3\}$ and $B = \mathbb{R} - \{-1\}$. Show that $f : A \to B$, $f(x) = \frac{-x-7}{x-3}$ is a bijection and find f^{-1}. [5]

One-one: $f(x_1) = f(x_2) \Rightarrow (-x_1 - 7)(x_2 - 3) = (-x_2 - 7)(x_1 - 3)$

$\Rightarrow 10(x_1 - x_2) = 0 \Rightarrow x_1 = x_2$, so f is one-one.

Onto: for $y \in B$, $y = \frac{-x-7}{x-3} \Rightarrow y(x-3) = -x - 7$

$\Rightarrow x = \frac{3y-7}{y+1}$, defined since $y \neq -1$, and $x \neq 3$ (else $3 = -7$).

So $x \in A$ and $f(x) = y$: f is onto. Hence f is a bijection.

$f^{-1}(x) = \frac{3x-7}{x+1}$

Ans: $f^{-1}(x) = \frac{3x-7}{x+1}$

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17. Find the domain of $f(x) = \sqrt{x^2 + 9x + 20} + \dfrac{1}{\sqrt{-x-2}}$. [5]

We need $x^2 + 9x + 20 \geq 0$ and $-x - 2 > 0$.

$x^2 + 9x + 20 = (x+4)(x+5) \geq 0$: the roots are $-5 < -4$ and $a > 0$.

$\Rightarrow x \in (-\infty, -5] \cup [-4, \infty)$

$-x - 2 > 0 \Rightarrow x < -2$

Common part: domain $= (-\infty, -5] \cup [-4, -2)$

Ans: $(-\infty, -5] \cup [-4, -2)$

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18. Find the range of $f(x) = \dfrac{x^2 - 3x + 3}{x^2 - 2x + 2}$, $x \in \mathbb{R}$. [5]

The denominator $x^2 - 2x + 2$ has $\Delta = -4 < 0$, so it is never zero: the domain is $\mathbb{R}$.

$y = \frac{x^2-3x+3}{x^2-2x+2} \Rightarrow y(x^2 - 2x + 2) = x^2 - 3x + 3$

$\Rightarrow (y-1)x^2 + (3-2y)x + (2y-3) = 0$

For $y \neq 1$, x is real, so $\Delta = (3-2y)^2 - 4(y-1)(2y-3) \geq 0$

$\Rightarrow -4y^2 + 8y - 3 \geq 0$, $\Rightarrow \frac{1}{2} \leq y \leq \frac{3}{2}$

The value $y = 1$ is taken at $x = 1$, and 1 lies in this interval.

Range $= [\frac{1}{2}, \frac{3}{2}]$

Ans: $[\frac{1}{2}, \frac{3}{2}]$

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19. The point $(9, -1)$ lies on $y = f(x)$. Which point must lie on $y = f(x - 4)$? [1]

The graph of $y = f(x - 4)$ is obtained by shift right by 4: $(x, y) \to (x + 4, y)$.

So $(9, -1) \to (13, -1)$.

Ans: $(13, -1)$

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20. Describe how to get the graph of $y = \sqrt{x + 1} - 1$ from $y = \sqrt{x}$, and find its domain and range. [3]

$\sqrt{x + 1}$: shift $y = \sqrt{x}$ left by 1.

Adding -1: shift down by 1.

$x + 1 \geq 0 \Rightarrow x \geq -1$

Domain $= [-1, \infty)$; since $\sqrt{} \geq 0$, range $= [-1, \infty)$.

Ans: Domain $[-1, \infty)$**, range** $[-1, \infty)$
