EXPRESSWALA ACADEMY

Maths 1A: Functions

Question Paper, Set 05

Time: 1 hour 30 minutes | Maximum marks: 50

Name: _____ Class: _____ Date: _____

1. Answer all 20 questions. The marks for each question are shown in brackets.
2. Show your working for the 2, 3 and 5 mark questions.
3. The key sheet and worked solutions are at the end of this paper.

1. If $A = \{1, 2, 7, 8, 9\}$ and $B = \{q, r, s, v, w\}$, find the number of relations from A to B. [1 mark]

2. If $f : A \to \mathbb{Z}$ is defined by $f(x) = x^2 + 1$ where $A = \{-3, -2, 0, 1, 4\}$, find the range of f. [1 mark]

3. Find the degree of the polynomial function $f(x) = -7x^5 + 4x^3 + 8x^2 + 6$. [1 mark]

4. Find the minimum value of $f(x) = x^2 + 2x + 4$. [2 marks]

5. Solve $x^2 - 4x + 8 \leq 0$. [2 marks]

6. Solve $|2x - 3| < 9$. [2 marks]

7. Find $[-4.8] + [4.8]$. [1 mark]

8. Solve $2[x] + 5 = [x] + 1$, where $[x]$ is the greatest integer function. [3 marks]

9. Find $\operatorname{sgn}(x^2 - 2x - 24)$ at $x = -\frac{1}{2}$. [1 mark]

10. If $f(x) = -3x - 5$ and $g(x) = x^2 + 3$, find $(g \circ f)(x)$. [2 marks]

11. If $f(x) = \dfrac{1 - 3x}{3x + 3}$, show that $(f \circ f)(x) = x$ and hence find $(f \circ f \circ f)(6)$. [5 marks]

12. Write $f(x) = 4x^3 - 6x^2 - 3x - 4$ as the sum of an even function and an odd function. [3 marks]

13. If $f(x) = x + 4$ and $g(x) = x^2 - 4$, find $(fg)(4)$ and $\left(\dfrac{f}{g}\right)(3)$. [2 marks]

14. f satisfies $f(x + y) = f(x)\, f(y)$ for all real x, y, and $f(1) = 5$. Find $f(3)$. [3 marks]

15. How many one-one functions are there from $A = \{1, 2, 6\}$ to $B = \{a, c, d, e, f, h\}$? [2 marks]

16. Let $A = \mathbb{R} - \{1\}$ and $B = \mathbb{R} - \left\{-\frac{3}{2}\right\}$. Show that $f : A \to B$, $f(x) = \frac{-3x-6}{2x-2}$ is a bijection and find f^{-1}. [5 marks]

17. Find the domain of $f(x) = \sqrt{x^2 - 7x + 10} + \dfrac{1}{\sqrt{8 - x}}$. [5 marks]

18. Find the range of $f(x) = \dfrac{x^2 - 3x + 4}{x^2 + x + 4}$, $x \in \mathbb{R}$. [5 marks]

19. The point $(3, -1)$ lies on $y = f(x)$. Which point must lie on $y = 3f(x)$? [1 mark]

20. Where does the graph of $y = |x^2 + 7x + 10|$ meet the x-axis, and what is the highest point of the arch between these points? [3 marks]

Key sheet

Q	Answer
1	33554432
2	$\text{Range} = \{1, 2, 5, 10, 17\}$
3	$\text{Degree} = 5$
4	3
5	ϕ
6	$x \in (-3, 6)$
7	-1
8	$x \in [-4, -3)$
9	-1
10	$(g \circ f)(x) = 9x^2 + 30x + 28$
11	$(f \circ f)(x) = x$; $(f \circ f \circ f)(6) = -\frac{17}{21}$
12	Even part $-6x^2 - 4$, odd part $4x^3 - 3x$
13	$(fg)(4) = 96$, $\left(\frac{f}{g}\right)(3) = \frac{7}{5}$
14	$f(3) = 125$
15	120
16	$f^{-1}(x) = \frac{2x-6}{2x+3}$
17	$(-\infty, 2] \cup [5, 8)$
18	$[\frac{1}{5}, \frac{7}{3}]$
19	$(3, -3)$
20	At $x = -5$ and $x = -2$; the top of the arch is $(-\frac{7}{2}, \frac{9}{4})$

Solutions

1. If $A = \{1, 2, 7, 8, 9\}$ and $B = \{q, r, s, v, w\}$, find the number of relations from A to B. [1]

$n(A \times B) = 5 \times 5 = 25$

Every subset of $A \times B$ is a relation.

$\text{Number of relations} = 2^{25} = 33554432$

Ans: 33554432

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2. If $f : A \to \mathbb{Z}$ is defined by $f(x) = x^2 + 1$ where $A = \{-3, -2, 0, 1, 4\}$, find the range of f. [1]

$f(-3) = 10$, $f(-2) = 5$, $f(0) = 1$, $f(1) = 2$, $f(4) = 17$

Range $= \{1, 2, 5, 10, 17\}$

Ans: $\text{Range} = \{1, 2, 5, 10, 17\}$

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3. Find the degree of the polynomial function $f(x) = -7x^5 + 4x^3 + 8x^2 + 6$. [1]

The degree is the highest power of x with a non-zero coefficient.

The highest power is x^5, so the degree is 5.

Ans: $\text{Degree} = 5$

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4. Find the minimum value of $f(x) = x^2 + 2x + 4$. [2]

$\frac{4ac - b^2}{4a} = \frac{4(1)(4) - (2)^2}{4(1)} = 3$

It occurs at $x = -\frac{b}{2a} = -1$.

Ans: 3

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5. Solve $x^2 - 4x + 8 \leq 0$. [2]

$\Delta = -16 < 0$

So the expression has the sign of $a = 1$ for every real x.

Solution set: ϕ

Ans: ϕ

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6. Solve $|2x - 3| < 9$. [2]

$-9 < 2x - 3 < 9$

Solution set: $(-3, 6)$

Ans: $x \in (-3, 6)$

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7. Find $[-4.8] + [4.8]$. [1]

$[-4.8] = -5$, $[4.8] = 4$

The sum is -1 (for a non-integer x, $[x] + [-x] = -1$).

Ans: -1

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8. Solve $2[x] + 5 = [x] + 1$, where $[x]$ is the greatest integer function. [3]

$1[x] = -4 \Rightarrow [x] = -4$

$[x] = -4 \iff -4 \le x < -3$

So $x \in [-4, -3)$.

Ans: $x \in [-4, -3)$

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9. Find $\operatorname{sgn}(x^2 - 2x - 24)$ at $x = -\frac{1}{2}$. [1]

$x^2 - 2x - 24 = -\frac{91}{4}$ at $x = -\frac{1}{2}$

This is negative, so sgn is -1.

Ans: -1

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10. If $f(x) = -3x - 5$ and $g(x) = x^2 + 3$, find $(g \circ f)(x)$. [2]

$(g \circ f)(x) = g(f(x)) = g(-3x - 5)$

$= (-3x - 5)^2 + 3$

$= 9x^2 + 30x + 28$

Ans: $(g \circ f)(x) = 9x^2 + 30x + 28$

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11. If $f(x) = \dfrac{1 - 3x}{3x + 3}$, show that $(f \circ f)(x) = x$ and hence find $(f \circ f \circ f)(6)$. [5]

$(f \circ f)(x) = f\left(\frac{1-3x}{3x+3}\right) = \dfrac{-3\left(\frac{1-3x}{3x+3}\right) + 1}{3\left(\frac{1-3x}{3x+3}\right) + 3}$

Multiply the numerator and the denominator by $3x + 3$:

$= \dfrac{(3x + 3) + (9x - 3)}{(3 - 9x) + (9x + 9)} = \dfrac{12x}{12}$

$= x$

So $f \circ f = I$, and $(f \circ f \circ f)(x) = f(x)$.

$(f \circ f \circ f)(6) = f(6) = \frac{-17}{21} = -\frac{17}{21}$

Ans: $(f \circ f)(x) = x$; $(f \circ f \circ f)(6) = -\frac{17}{21}$

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12. Write $f(x) = 4x^3 - 6x^2 - 3x - 4$ as the sum of an even function and an odd function. [3]

$f(x) = \frac{1}{2}[f(x) + f(-x)] + \frac{1}{2}[f(x) - f(-x)]$

$f(-x) = -4x^3 - 6x^2 + 3x - 4$

$\frac{1}{2}[f(x) + f(-x)] = -6x^2 - 4$ (even)

$\frac{1}{2}[f(x) - f(-x)] = 4x^3 - 3x$ (odd)

$f(x) = \left(-6x^2 - 4\right) + \left(4x^3 - 3x\right)$

Ans: Even part $-6x^2 - 4$, odd part $4x^3 - 3x$

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13. If $f(x) = x + 4$ and $g(x) = x^2 - 4$, find $(fg)(4)$ and $\left(\dfrac{f}{g}\right)(3)$. [2]

$(fg)(4) = f(4)\, g(4) = (8)(12) = 96$

$\left(\frac{f}{g}\right)(3) = \frac{f(3)}{g(3)} = \frac{7}{5} = \frac{7}{5}$

Ans: $(fg)(4) = 96$, $\left(\frac{f}{g}\right)(3) = \frac{7}{5}$

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14. f satisfies $f(x+y) = f(x)\,f(y)$ for all real x, y, and $f(1) = 5$. Find $f(3)$. [3]

$f(3) = f(1+1+\cdots+1) = f(1)^3$

$= (5)^3 = 125$

Ans: $f(3) = 125$

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15. How many one-one functions are there from $A = \{1, 2, 6\}$ to $B = \{a, c, d, e, f, h\}$? [2]

$n(A) = 3 \le n(B) = 6$

$\text{Number} = {}^6P_3 = 120$

Ans: 120

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16. Let $A = \mathbb{R} - \{1\}$ and $B = \mathbb{R} - \left\{-\frac{3}{2}\right\}$. Show that $f : A \to B$, $f(x) = \frac{-3x-6}{2x-2}$ is a bijection and find f^{-1}. [5]

One-one: $f(x_1) = f(x_2) \Rightarrow (-3x_1 - 6)(2x_2 - 2) = (-3x_2 - 6)(2x_1 - 2)$

$\Rightarrow 18(x_1 - x_2) = 0 \Rightarrow x_1 = x_2$, so f is one-one.

Onto: for $y \in B$, $y = \frac{-3x-6}{2x-2} \Rightarrow y(2x-2) = -3x - 6$

$\Rightarrow x = \frac{2y-6}{2y+3}$, defined since $y \neq -\frac{3}{2}$, and $x \neq 1$ (else $6 = -12$).

So $x \in A$ and $f(x) = y$: f is onto. Hence f is a bijection.

$f^{-1}(x) = \frac{2x-6}{2x+3}$

Ans: $f^{-1}(x) = \frac{2x-6}{2x+3}$

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17. Find the domain of $f(x) = \sqrt{x^2 - 7x + 10} + \dfrac{1}{\sqrt{8-x}}$. [5]

We need $x^2 - 7x + 10 \ge 0$ and $8 - x > 0$.

$x^2 - 7x + 10 = (x-5)(x-2) \ge 0$: the roots are $2 < 5$ and $a > 0$.

$\Rightarrow x \in (-\infty, 2] \cup [5, \infty)$

$8 - x > 0 \Rightarrow x < 8$

Common part: domain $= (-\infty, 2] \cup [5, 8)$

Ans: $(-\infty, 2] \cup [5, 8)$

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18. Find the range of $f(x) = \dfrac{x^2 - 3x + 4}{x^2 + x + 4}$, $x \in \mathbb{R}$. [5]

The denominator $x^2 + x + 4$ has $\Delta = -15 < 0$, so it is never zero: the domain is $\mathbb{R}$.

$y = \frac{x^2-3x+4}{x^2+x+4} \Rightarrow y(x^2 + x + 4) = x^2 - 3x + 4$

$\Rightarrow (y-1)x^2 + (y+3)x + (4y-4) = 0$

For $y \neq 1$, x is real, so $\Delta = (y+3)^2 - 4(y-1)(4y-4) \ge 0$

$\Rightarrow -15y^2 + 38y - 7 \ge 0$, $\Rightarrow \frac{1}{5} \le y \le \frac{7}{3}$

The value $y = 1$ is taken at $x = 0$, and 1 lies in this interval.

Range $= [\frac{1}{5}, \frac{7}{3}]$

Ans: $[\frac{1}{5}, \frac{7}{3}]$

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19. The point $(3, -1)$ lies on $y = f(x)$. Which point must lie on $y = 3f(x)$? [1]

The graph of $y = 3f(x)$ is obtained by vertical stretch: $(x, y) \to (x, 3y)$.

So $(3, -1) \to (3, -3)$.

Ans: $(3, -3)$

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20. Where does the graph of $y = |x^2 + 7x + 10|$ meet the x-axis, and what is the highest point of the arch between these points? [3]

$y = x^2 + 7x + 10$ cuts the x-axis where $(x + 2)(x + 5) = 0$: $x = -5, -2$.
Between them it dips to $-\frac{9}{4}$ at $x = -\frac{7}{2}$.
Taking $|f(x)|$ flips this part up: the arch has top $(-\frac{7}{2}, \frac{9}{4})$.
Ans: At $x = -5$ and $x = -2$; the top of the arch is $(-\frac{7}{2}, \frac{9}{4})$