EXPRESSWALA ACADEMY

Maths 1A: Functions

Question Paper, Set 03

Time: 1 hour 30 minutes | Maximum marks: 50

Name: _____ Class: _____ Date: _____

1. Answer all 20 questions. The marks for each question are shown in brackets.
2. Show your working for the 2, 3 and 5 mark questions.
3. The key sheet and worked solutions are at the end of this paper.

1. If $A = \{1, 7\}$ and $B = \{r, s, t, v, w\}$, how many elements does $A \times B$ have? [1 mark]

2. If $f : A \to \mathbb{Z}$ is defined by $f(x) = x^2 + 4$ where $A = \{-2, 1, 2, 3, 4\}$, find the range of f. [1 mark]

3. Find the degree of the polynomial function $f(x) = -7x^2 + 5x - 4x^6 + 9x^7$. [1 mark]

4. Find the minimum value of $f(x) = 2x^2 + 12x + 25$. [2 marks]

5. Solve $-x^2 - 2x + 3 \geq 0$. [2 marks]

6. Solve $|x - 2| > 8$. [2 marks]

7. Find the fractional part $\{2.6\}$. [1 mark]

8. Solve $[x]^2 - 3[x] - 10 = 0$, where $[x]$ is the greatest integer function. [3 marks]

9. Find $\log_{27} 27$. [1 mark]

10. If $f(x) = 2 - 3x$ and $g(x) = x^2 - 3$, find $(g \circ f)(x)$. [2 marks]

11. If $f(x) = \dfrac{-2x - 5}{2x + 2}$, show that $(f \circ f)(x) = x$ and hence find $(f \circ f \circ f)(5)$. [5 marks]

12. Write $f(x) = -5x^3 + 3x^2 + 3x - 3$ as the sum of an even function and an odd function. [3 marks]

13. If $f(x) = x + 6$ and $g(x) = x^2 + 6x + 8$, find the domain of $\dfrac{f}{g}$. [2 marks]

14. A polynomial f satisfies $f(x)f\left(\frac{1}{x}\right) = f(x) + f\left(\frac{1}{x}\right)$ and $f(5) = -624$. Find $f(2)$. [3 marks]

15. How many bijections are there from $A = \{1, 2, 7, 8\}$ to $B = \{a, b, d, g\}$? [2 marks]

16. Let $A = \mathbb{R} - \{2\}$ and $B = \mathbb{R} - \left\{\frac{4}{3}\right\}$. Show that $f : A \to B$, $f(x) = \frac{4x-2}{3x-6}$ is a bijection and find f^{-1}. [5 marks]

17. Find the domain of $f(x) = \sqrt{x^2 - 16} + \log_e(x - 8)$. [5 marks]

18. Find the range of $f(x) = \dfrac{x^2 + x + 4}{x^2 + 3}$, $x \in \mathbb{R}$. [5 marks]

19. The point $(4, 7)$ lies on $y = f(x)$. Which point must lie on $y = f(-x)$? [1 mark]

20. Where does the graph of $y = |x^2 - x - 12|$ meet the x-axis, and what is the highest point of the arch between these points? [3 marks]

Key sheet

Q	Answer
1	10
2	$\text{Range} = \{5, 8, 13, 20\}$
3	$\text{Degree} = 7$
4	7
5	$x \in [-3, 1]$
6	$x \in (-\infty, -6) \cup (10, \infty)$
7	$\frac{3}{5}$
8	$x \in [-2, -1) \cup [5, 6)$
9	1
10	$(g \circ f)(x) = 9x^2 - 12x + 1$
11	$(f \circ f)(x) = x$; $(f \circ f \circ f)(5) = -\frac{5}{4}$
12	Even part $3x^2 - 3$, odd part $-5x^3 + 3x$
13	$\mathbb{R} - \{-4, -2\}$
14	$f(2) = -15$
15	24
16	$f^{-1}(x) = \frac{6x-2}{3x-4}$
17	$(8, \infty)$
18	$[\frac{5}{6}, \frac{3}{2}]$
19	$(-4, 7)$
20	At $x = -3$ and $x = 4$; the top of the arch is $(\frac{1}{2}, \frac{49}{4})$

Solutions

1. If $A = \{1, 7\}$ and $B = \{r, s, t, v, w\}$, how many elements does $A \times B$ have? [1]

$n(A) = 2,\ n(B) = 5$

$n(A \times B) = n(A) \cdot n(B) = 10$

Ans: 10

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2. If $f : A \to \mathbb{Z}$ is defined by $f(x) = x^2 + 4$ where $A = \{-2, 1, 2, 3, 4\}$, find the range of f. [1]

$f(-2) = 8,\ f(1) = 5,\ f(2) = 8,\ f(3) = 13,\ f(4) = 20$

Range $= \{5, 8, 13, 20\}$

Ans: $\text{Range} = \{5, 8, 13, 20\}$

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3. Find the degree of the polynomial function $f(x) = -7x^2 + 5x - 4x^6 + 9x^7$. [1]

The degree is the highest power of x with a non-zero coefficient.

The highest power is x^7, so the degree is 7.

Ans: $\text{Degree} = 7$

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4. Find the minimum value of $f(x) = 2x^2 + 12x + 25$. [2]

$\frac{4ac - b^2}{4a} = \frac{4(2)(25) - (12)^2}{4(2)} = 7$

It occurs at $x = -\frac{b}{2a} = -3$.

Ans: 7

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5. Solve $-x^2 - 2x + 3 \geq 0$. [2]

$-x^2 - 2x + 3 = -(x - 1)(x + 3)$

The roots are -3 and 1, and $a = -1 < 0$.

The expression has the sign opposite to a between the roots.

Solution set: $[-3, 1]$

Ans: $x \in [-3, 1]$

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6. Solve $|x - 2| > 8$. [2]

$x - 2 > 8$ or $x - 2 < -8$

Solution set: $(-\infty, -6) \cup (10, \infty)$

Ans: $x \in (-\infty, -6) \cup (10, \infty)$

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7. Find the fractional part $\{2.6\}$. [1]

$2.6 = 2 + 0.6$

$\{2.6\} = 2.6 - (2) = 0.6$

Ans: $\frac{3}{5}$

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8. Solve $[x]^2 - 3[x] - 10 = 0$, where $[x]$ is the greatest integer function. [3]

Put $t = [x]$: $t^2 - 3t - 10 = 0$

$(t+2)(t-5) = 0 \Rightarrow t = -2$ or 5

$[x] = -2 \Rightarrow x \in [-2, -1)$, $[x] = 5 \Rightarrow x \in [5, 6)$

So $x \in [-2, -1) \cup [5, 6)$.

Ans: $x \in [-2, -1) \cup [5, 6)$

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9. Find $\log_{27} 27$. [1]

$\log_{27} 27 = \log_{3^3} 3^3$

$= \frac{3}{3} \log_3 3 = 1$

Ans: 1

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10. If $f(x) = 2 - 3x$ and $g(x) = x^2 - 3$, find $(g \circ f)(x)$. [2]

$(g \circ f)(x) = g(f(x)) = g(2 - 3x)$

$= (2 - 3x)^2 - 3$

$= 9x^2 - 12x + 1$

Ans: $(g \circ f)(x) = 9x^2 - 12x + 1$

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11. If $f(x) = \dfrac{-2x - 5}{2x + 2}$, show that $(f \circ f)(x) = x$ and hence find $(f \circ f \circ f)(5)$. [5]

$(f \circ f)(x) = f\left(\frac{-2x-5}{2x+2}\right) = \dfrac{-2\left(\frac{-2x-5}{2x+2}\right) - 5}{2\left(\frac{-2x-5}{2x+2}\right) + 2}$

Multiply the numerator and the denominator by $2x + 2$:

$= \dfrac{(-10x - 10) + (4x + 10)}{(-4x - 10) + (4x + 4)} = \dfrac{-6x}{-6}$

$= x$

So $f \circ f = I$, and $(f \circ f \circ f)(x) = f(x)$.

$(f \circ f \circ f)(5) = f(5) = \frac{-15}{12} = -\frac{5}{4}$

Ans: $(f \circ f)(x) = x$; $(f \circ f \circ f)(5) = -\frac{5}{4}$

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12. Write $f(x) = -5x^3 + 3x^2 + 3x - 3$ as the sum of an even function and an odd function. [3]

$f(x) = \frac{1}{2}[f(x) + f(-x)] + \frac{1}{2}[f(x) - f(-x)]$

$f(-x) = 5x^3 + 3x^2 - 3x - 3$

$\frac{1}{2}[f(x) + f(-x)] = 3x^2 - 3$ (even)

$\frac{1}{2}[f(x) - f(-x)] = -5x^3 + 3x$ (odd)

$f(x) = \left(3x^2 - 3\right) + \left(-5x^3 + 3x\right)$

Ans: Even part $3x^2 - 3$, odd part $-5x^3 + 3x$

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13. If $f(x) = x + 6$ and $g(x) = x^2 + 6x + 8$, find the domain of $\dfrac{f}{g}$. [2]

$D_f = D_g = \mathbb{R}$, and $\frac{f}{g}$ needs $g(x) \neq 0$.

$(x + 2)(x + 4) = 0 \Rightarrow x = -4, -2$

Domain $= \mathbb{R} - \{-4, -2\}$

Ans: $\mathbb{R} - \{-4, -2\}$

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14. A polynomial f satisfies $f(x)f\left(\frac{1}{x}\right) = f(x) + f\left(\frac{1}{x}\right)$ and $f(5) = -624$. Find $f(2)$. [3]

So $f(x) = 1 \pm x^n$.

$1 - 5^n = -624 \Rightarrow 5^n = 625 \Rightarrow n = 4$

$f(x) = 1 - x^4$

$f(2) = 1 - (2)^4 = -15$

Ans: $f(2) = -15$

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15. How many bijections are there from $A = \{1, 2, 7, 8\}$ to $B = \{a, b, d, g\}$? [2]

$n(A) = n(B) = 4$

$\text{Number of bijections} = 4! = 24$

Ans: 24

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16. Let $A = \mathbb{R} - \{2\}$ and $B = \mathbb{R} - \left\{\frac{4}{3}\right\}$. Show that $f : A \to B$, $f(x) = \frac{4x-2}{3x-6}$ is a bijection and find f^{-1}. [5]

One-one: $f(x_1) = f(x_2) \Rightarrow (4x_1 - 2)(3x_2 - 6) = (4x_2 - 2)(3x_1 - 6)$

$\Rightarrow -18(x_1 - x_2) = 0 \Rightarrow x_1 = x_2$, so f is one-one.

Onto: for $y \in B$, $y = \frac{4x-2}{3x-6} \Rightarrow y(3x - 6) = 4x - 2$

$\Rightarrow x = \frac{6y-2}{3y-4}$, defined since $y \neq \frac{4}{3}$, and $x \neq 2$ (else $-24 = -6$).

So $x \in A$ and $f(x) = y$: f is onto. Hence f is a bijection.

$f^{-1}(x) = \frac{6x-2}{3x-4}$

Ans: $f^{-1}(x) = \frac{6x-2}{3x-4}$

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17. Find the domain of $f(x) = \sqrt{x^2 - 16} + \log_e(x - 8)$. [5]

We need $x^2 - 16 \geq 0$ and $x - 8 > 0$.

$x^2 - 16 = (x - 4)(x + 4) \geq 0$: the roots are $-4 < 4$ and $a > 0$.

$\Rightarrow x \in (-\infty, -4] \cup [4, \infty)$

$x - 8 > 0 \Rightarrow x > 8$

Common part: domain $= (8, \infty)$

Ans: $(8, \infty)$

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18. Find the range of $f(x) = \dfrac{x^2 + x + 4}{x^2 + 3}$, $x \in \mathbb{R}$. [5]

The denominator $x^2 + 3$ has $\Delta = -12 < 0$, so it is never zero: the domain is $\mathbb{R}$.

$y = \frac{x^2+x+4}{x^2+3} \Rightarrow y(x^2 + 3) = x^2 + x + 4$

$\Rightarrow (y - 1)x^2 + (-1)x + (3y - 4) = 0$

For $y \neq 1$, x is real, so $\Delta = (-1)^2 - 4(y - 1)(3y - 4) \geq 0$

$\Rightarrow -12y^2 + 28y - 15 \geq 0, \Rightarrow \frac{5}{6} \leq y \leq \frac{3}{2}$

The value $y = 1$ is taken at $x = -1$, and 1 lies in this interval.

Range $= [\frac{5}{6}, \frac{3}{2}]$

Ans: $[\frac{5}{6}, \frac{3}{2}]$

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19. The point $(4, 7)$ lies on $y = f(x)$. Which point must lie on $y = f(-x)$? [1]

The graph of $y = f(-x)$ is obtained by reflection in the y-axis: $(x, y) \to (-x, y)$.

So $(4, 7) \to (-4, 7)$.

Ans: $(-4, 7)$

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20. Where does the graph of $y = |x^2 - x - 12|$ meet the x-axis, and what is the highest point of the arch between these points? [3]

$y = x^2 - x - 12$ cuts the x-axis where $(x - 4)(x + 3) = 0$: $x = -3, 4$.

Between them it dips to $-\frac{49}{4}$ at $x = \frac{1}{2}$.

Taking $|f(x)|$ flips this part up: the arch has top $(\frac{1}{2}, \frac{49}{4})$.

Ans: At $x = -3$ and $x = 4$; the top of the arch is $(\frac{1}{2}, \frac{49}{4})$
