EXPRESSWALA ACADEMY

Maths 1A: Functions

Question Paper, Set 02

Time: 1 hour 30 minutes | Maximum marks: 50

Name: _____ Class: _____ Date: _____

1. Answer all 20 questions. The marks for each question are shown in brackets.
2. Show your working for the 2, 3 and 5 mark questions.
3. The key sheet and worked solutions are at the end of this paper.

1. If $A = \{2, 4, 6, 8\}$ and $B = \{t, v\}$, how many functions are there from B to A? [1 mark]

2. If $f : A \to \mathbb{Z}$ is defined by $f(x) = x^2 - 3$ where $A = \{-2, 1, 2, 3\}$, find the range of f. [1 mark]

3. Find the point where the line $y = -5x - 4$ meets the y-axis. [1 mark]

4. Find the minimum value of $f(x) = 2x^2 - 16x + 38$. [2 marks]

5. Solve $12 - 3x^2 < 0$. [2 marks]

6. Solve $|2x - 3| > 3$. [2 marks]

7. Find $[5.4] + [-5.4]$. [1 mark]

8. Solve $3[x] + 2 = [x] + 6$, where $[x]$ is the greatest integer function. [3 marks]

9. Find the value of $9^{2\log_9 9}$. [1 mark]

10. If $f(x) = -x - 5$, find $(f \circ f)(x)$. [2 marks]

11. If $f(x) = \dfrac{x+3}{x-1}$, show that $(f \circ f)(x) = x$ and hence find $(f \circ f \circ f)(5)$. [5 marks]

12. Write $f(x) = 2^x$ as the sum of an even function and an odd function. [3 marks]

13. If $f(x) = x + 1$ and $g(x) = x^2 - 2x - 24$, find the domain of $\dfrac{f}{g}$. [2 marks]

14. f satisfies $f(x + y) = f(x) + f(y)$ for all real x, y, and $f(3) = 12$. Find $f(2)$. [3 marks]

15. How many bijections are there from $A = \{1, 8, 9, 10, 11, 12\}$ to $B = \{a, b, c, e, f, h\}$? [2 marks]

16. Show that $f : \mathbb{R} \to \mathbb{R}$, $f(x) = \frac{3-2x}{5}$ is a bijection and find f^{-1}. [5 marks]

17. Find the domain of $f(x) = \sqrt{x^2 - 7x + 10} + \dfrac{1}{\sqrt{9 - x}}$. [5 marks]

18. Find the range of $f(x) = \dfrac{x^2 + 2x + 5}{x^2 + 3x + 6}$, $x \in \mathbb{R}$. [5 marks]

19. The point $(3, -8)$ lies on $y = f(x)$. Which point must lie on $y = f(x) + 3$? [1 mark]

20. Where does the graph of $y = |x^2 - x - 2|$ meet the x-axis, and what is the highest point of the arch between these points? [3 marks]

Key sheet

Q	Answer
1	16
2	$\text{Range} = \{-2, 1, 6\}$
3	$(0, -4)$
4	6
5	$x \in (-\infty, -2) \cup (2, \infty)$
6	$x \in (-\infty, 0) \cup (3, \infty)$
7	-1
8	$x \in [2, 3)$
9	81
10	$(f \circ f)(x) = x$
11	$(f \circ f)(x) = x$; $(f \circ f \circ f)(5) = 2$
12	Even part $\frac{2^x}{2} + \frac{2^{-x}}{2}$, odd part $\frac{2^x}{2} - \frac{2^{-x}}{2}$
13	$\mathbb{R} - \{-4, 6\}$
14	$f(2) = 8$
15	720
16	$f^{-1}(x) = \frac{3}{2} - \frac{5x}{2}$
17	$(-\infty, 2] \cup [5, 9)$
18	$[\frac{4}{5}, \frac{4}{3}]$
19	$(3, -5)$
20	At $x = -1$ and $x = 2$; the top of the arch is $(\frac{1}{2}, \frac{9}{4})$

Solutions

1. If $A = \{2, 4, 6, 8\}$ and $B = \{t, v\}$, how many functions are there from B to A? [1]

$n(B) = 2,\ n(A) = 4$

Number of functions from B to $A = \{n(A)\}^{n(B)} = 4^2 = 16$

Ans: 16

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2. If $f : A \to \mathbb{Z}$ is defined by $f(x) = x^2 - 3$ where $A = \{-2, 1, 2, 3\}$, find the range of f. [1]

$f(-2) = 1,\ f(1) = -2,\ f(2) = 1,\ f(3) = 6$

Range $= \{-2, 1, 6\}$

Ans: $\text{Range} = \{-2, 1, 6\}$

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3. Find the point where the line $y = -5x - 4$ meets the y-axis. [1]

$x = 0 \Rightarrow y = -4$

The point is $(0, -4)$.

Ans: $(0, -4)$

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4. Find the minimum value of $f(x) = 2x^2 - 16x + 38$. [2]

$\frac{4ac - b^2}{4a} = \frac{4(2)(38) - (-16)^2}{4(2)} = 6$

It occurs at $x = -\frac{b}{2a} = 4$.

Ans: 6

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5. Solve $12 - 3x^2 < 0$. [2]

$12 - 3x^2 = -3\,(x - 2)\,(x + 2)$

The roots are -2 and 2, and $a = -3 < 0$.

The expression has the sign of a outside the roots.

Solution set: $(-\infty, -2) \cup (2, \infty)$

Ans: $x \in (-\infty, -2) \cup (2, \infty)$

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6. Solve $|2x - 3| > 3$. [2]

$2x - 3 > 3$ or $2x - 3 < -3$

Solution set: $(-\infty, 0) \cup (3, \infty)$

Ans: $x \in (-\infty, 0) \cup (3, \infty)$

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7. Find $[5.4] + [-5.4]$. [1]

$[5.4] = 5,\ [-5.4] = -6$

The sum is -1 (for a non-integer x, $[x] + [-x] = -1$).

Ans: -1

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8. Solve $3[x] + 2 = [x] + 6$, where $[x]$ is the greatest integer function. [3]

$2[x] = 4 \Rightarrow [x] = 2$

$[x] = 2 \iff 2 \le x < 3$

So $x \in [2, 3)$.

Ans: $x \in [2, 3)$

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9. Find the value of $9^{2\log_9 9}$. [1]

$9^{2\log_9 9} = 9^{\log_9 9^2}$

$= 9^2 = 81$ (using $a^{\log_a N} = N$)

Ans: 81

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10. If $f(x) = -x - 5$, find $(f \circ f)(x)$. [2]

$(f \circ f)(x) = f(-x - 5) = -1(-x - 5) - 5$

$= x$

Ans: $(f \circ f)(x) = x$

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11. If $f(x) = \dfrac{x+3}{x-1}$, show that $(f \circ f)(x) = x$ and hence find $(f \circ f \circ f)(5)$. [5]

$(f \circ f)(x) = f\left(\frac{x+3}{x-1}\right) = \dfrac{1\left(\frac{x+3}{x-1}\right) + 3}{1\left(\frac{x+3}{x-1}\right) - 1}$

Multiply the numerator and the denominator by $x - 1$:

$= \dfrac{(x+3) + (3x-3)}{(1-x) + (x+3)} = \dfrac{4x}{4}$

$= x$

So $f \circ f = I$, and $(f \circ f \circ f)(x) = f(x)$.

$(f \circ f \circ f)(5) = f(5) = \frac{8}{4} = 2$

Ans: $(f \circ f)(x) = x$; $(f \circ f \circ f)(5) = 2$

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12. Write $f(x) = 2^x$ as the sum of an even function and an odd function. [3]

$f(x) = \frac{1}{2}[f(x) + f(-x)] + \frac{1}{2}[f(x) - f(-x)]$

$f(-x) = 2^{-x}$

$\frac{1}{2}[f(x) + f(-x)] = \frac{2^x}{2} + \frac{2^{-x}}{2}$ (even)

$\frac{1}{2}[f(x) - f(-x)] = \frac{2^x}{2} - \frac{2^{-x}}{2}$ (odd)

$f(x) = \left(\frac{2^x}{2} + \frac{2^{-x}}{2}\right) + \left(\frac{2^x}{2} - \frac{2^{-x}}{2}\right)$

Ans: Even part $\frac{2^x}{2} + \frac{2^{-x}}{2}$, odd part $\frac{2^x}{2} - \frac{2^{-x}}{2}$

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13. If $f(x) = x + 1$ and $g(x) = x^2 - 2x - 24$, find the domain of $\dfrac{f}{g}$. [2]

$D_f = D_g = \mathbb{R}$, and $\frac{f}{g}$ needs $g(x) \neq 0$.

$(x - 6)(x + 4) = 0 \Rightarrow x = -4, 6$

Domain $= \mathbb{R} - \{-4, 6\}$

Ans: $\mathbb{R} - \{-4, 6\}$

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14. f satisfies $f(x+y) = f(x) + f(y)$ for all real x, y, and $f(3) = 12$. Find $f(2)$. [3]

$f(x+y) = f(x) + f(y) \Rightarrow f(x) = kx$.

$f(3) = 12 \Rightarrow 3k = 12 \Rightarrow k = 4$

$f(2) = 4 \times 2 = 8$

Ans: $f(2) = 8$

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15. How many bijections are there from $A = \{1, 8, 9, 10, 11, 12\}$ to $B = \{a, b, c, e, f, h\}$? [2]

$n(A) = n(B) = 6$

$\text{Number of bijections} = 6! = 720$

Ans: 720

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16. Show that $f : \mathbb{R} \to \mathbb{R}$, $f(x) = \frac{3-2x}{5}$ is a bijection and find f^{-1}. [5]

One-one: $f(x_1) = f(x_2) \Rightarrow \frac{3-2x_1}{5} = \frac{3-2x_2}{5}$

$\Rightarrow -2x_1 = -2x_2 \Rightarrow x_1 = x_2$, so f is one-one.

Onto: let $y \in \mathbb{R}$ and $y = \frac{3-2x}{5}$.

$\Rightarrow x = \frac{3}{2} - \frac{5y}{2} \in \mathbb{R}$

and $f(x) = y$, so every y has a pre-image: f is onto.

Hence f is a bijection, and $f^{-1}(x) = \frac{3}{2} - \frac{5x}{2}$

Ans: $f^{-1}(x) = \frac{3}{2} - \frac{5x}{2}$

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17. Find the domain of $f(x) = \sqrt{x^2 - 7x + 10} + \dfrac{1}{\sqrt{9-x}}$. [5]

We need $x^2 - 7x + 10 \geq 0$ and $9 - x > 0$.

$x^2 - 7x + 10 = (x-5)(x-2) \geq 0$: the roots are $2 < 5$ and $a > 0$.

$\Rightarrow x \in (-\infty, 2] \cup [5, \infty)$

$9 - x > 0 \Rightarrow x < 9$

Common part: domain $= (-\infty, 2] \cup [5, 9)$

Ans: $(-\infty, 2] \cup [5, 9)$

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18. Find the range of $f(x) = \dfrac{x^2 + 2x + 5}{x^2 + 3x + 6}$, $x \in \mathbb{R}$. [5]

The denominator $x^2 + 3x + 6$ has $\Delta = -15 < 0$, so it is never zero: the domain is $\mathbb{R}$.

$y = \frac{x^2+2x+5}{x^2+3x+6} \Rightarrow y(x^2 + 3x + 6) = x^2 + 2x + 5$

$\Rightarrow (y-1)x^2 + (3y-2)x + (6y-5) = 0$

For $y \neq 1$, x is real, so $\Delta = (3y-2)^2 - 4(y-1)(6y-5) \geq 0$

$\Rightarrow -15y^2 + 32y - 16 \geq 0$, $\Rightarrow \frac{4}{5} \leq y \leq \frac{4}{3}$

The value $y = 1$ is taken at $x = -1$, and 1 lies in this interval.

Range $= [\frac{4}{5}, \frac{4}{3}]$

Ans: $[\frac{4}{5}, \frac{4}{3}]$

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19. The point $(3, -8)$ lies on $y = f(x)$. Which point must lie on $y = f(x) + 3$? [1]

The graph of $y = f(x) + 3$ is obtained by shift up by 3: $(x, y) \to (x, y+3)$.

So $(3, -8) \to (3, -5)$.

Ans: $(3, -5)$

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20. Where does the graph of $y = |x^2 - x - 2|$ meet the x-axis, and what is the highest point of the arch between these points? [3]

$y = x^2 - x - 2$ cuts the x-axis where $(x - 2)(x + 1) = 0$: $x = -1, 2$.
Between them it dips to $-\frac{9}{4}$ at $x = \frac{1}{2}$.
Taking $|f(x)|$ flips this part up: the arch has top $(\frac{1}{2}, \frac{9}{4})$.
Ans: At $x = -1$ and $x = 2$; the top of the arch is $(\frac{1}{2}, \frac{9}{4})$