EXPRESSWALA ACADEMY

Maths 1A: Functions

Question Paper, Set 01

Time: 1 hour 30 minutes | Maximum marks: 50

Name: _____ Class: _____ Date: _____

1. Answer all 20 questions. The marks for each question are shown in brackets.
2. Show your working for the 2, 3 and 5 mark questions.
3. The key sheet and worked solutions are at the end of this paper.

1. If $A = \{2, 5\}$ and $B = \{p, t, u, w\}$, how many elements does $A \times B$ have? [1 mark]

2. Find the range of $f : A \to \mathbb{Z}$, $f(x) = |2x + 2|$, where $A = \{-3, -2, -1, 1, 2\}$. [1 mark]

3. How many constant functions are there from a set with 2 elements to a set with 3 elements? [1 mark]

4. Find the maximum value of $f(x) = -x^2 + 8x - 14$. [2 marks]

5. Solve $-x^2 + 3x - 2 < 0$. [2 marks]

6. Solve $|3x + 4| \geq 9$. [2 marks]

7. Find $\lceil 6.3 \rceil - \lceil -8.4 \rceil$. [1 mark]

8. Using $[x] + \left[x + \frac{1}{n}\right] + \cdots + \left[x + \frac{n-1}{n}\right] = [nx]$, evaluate $[0.46] + \left[0.46 + \frac{1}{5}\right] + \left[0.46 + \frac{2}{5}\right] + \left[0.46 + \frac{3}{5}\right] + \left[0.46 + \frac{4}{5}\right]$. [3 marks]

9. Find $\log_4 8$. [1 mark]

10. If $f(x) = 2x + 1$, find $(f \circ f)(x)$. [2 marks]

11. If $f(x) = \dfrac{3x - 5}{2x - 3}$, show that $(f \circ f)(x) = x$ and hence find $(f \circ f \circ f)(-3)$. [5 marks]

12. Write $f(x) = -2x^4 + 2x^3 + 4x^2 + 3x - 2$ as the sum of an even function and an odd function. [3 marks]

13. If $f(x) = x - 3$ and $g(x) = x^2 + 6x + 5$, find the domain of $\dfrac{f}{g}$. [2 marks]

14. f satisfies $f(x + y) = f(x) + f(y)$ for all real x, y, and $f(2) = 4$. Find $f(3)$. [3 marks]

15. How many onto functions are there from $A = \{1, 6, 7, 10, 11\}$ to $B = \{c, e\}$? [2 marks]

16. Let $A = \mathbb{R} - \{5\}$ and $B = \mathbb{R} - \{6\}$. Show that $f : A \to B$, $f(x) = \frac{6x+2}{x-5}$ is a bijection and find f^{-1}. [5 marks]

17. Find the domain of $f(x) = \sqrt{x^2 - 9x + 20} + \dfrac{1}{x - 1}$. [5 marks]

18. Find the range of $f(x) = \dfrac{x^2 + 3}{x^2 + 2x + 5}$, $x \in \mathbb{R}$. [5 marks]

19. The point $(2, 7)$ lies on $y = f(x)$. Which point must lie on $y = 5f(x)$? [1 mark]

20. Describe how to get the graph of $y = \sqrt{x - 2} + 4$ from $y = \sqrt{x}$, and find its domain and range. [3 marks]

Key sheet

Q	Answer
1	8
2	$\text{Range} = \{0, 2, 4, 6\}$
3	3
4	2
5	$x \in (-\infty, 1) \cup (2, \infty)$
6	$x \in (-\infty, -\frac{13}{3}] \cup [\frac{5}{3}, \infty)$
7	16
8	2
9	$\frac{3}{2}$
10	$(f \circ f)(x) = 4x + 3$
11	$(f \circ f)(x) = x$; $(f \circ f \circ f)(-3) = \frac{14}{9}$
12	Even part $-2x^4 + 4x^2 - 2$, odd part $2x^3 + 3x$
13	$\mathbb{R} - \{-5, -1\}$
14	$f(3) = 6$
15	30
16	$f^{-1}(x) = \frac{5x+2}{x-6}$
17	$(-\infty, 1) \cup (1, 4] \cup [5, \infty)$
18	$[\frac{1}{2}, \frac{3}{2}]$
19	$(2, 35)$
20	Domain $[2, \infty)$, range $[4, \infty)$

Solutions

1. If $A = \{2, 5\}$ and $B = \{p, t, u, w\}$, how many elements does $A \times B$ have? [1]

$n(A) = 2,\ n(B) = 4$

$n(A \times B) = n(A) \cdot n(B) = 8$

Ans: 8

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2. Find the range of $f : A \to \mathbb{Z}$, $f(x) = |2x + 2|$, where $A = \{-3, -2, -1, 1, 2\}$. [1]

$f(-3) = 4$, $f(-2) = 2$, $f(-1) = 0$, $f(1) = 4$, $f(2) = 6$

Range $= \{0, 2, 4, 6\}$

Ans: $\text{Range} = \{0, 2, 4, 6\}$

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3. How many constant functions are there from a set with 2 elements to a set with 3 elements? [1]

A constant function sends every element to the same value k.

There is one constant function for each element of the co-domain: 3.

Ans: 3

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4. Find the maximum value of $f(x) = -x^2 + 8x - 14$. [2]

$\frac{4ac - b^2}{4a} = \frac{4(-1)(-14) - (8)^2}{4(-1)} = 2$

It occurs at $x = -\frac{b}{2a} = 4$.

Ans: 2

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5. Solve $-x^2 + 3x - 2 < 0$. [2]

$-x^2 + 3x - 2 = -(x - 2)(x - 1)$

The roots are 1 and 2, and $a = -1 < 0$.

The expression has the sign of a outside the roots.

Solution set: $(-\infty, 1) \cup (2, \infty)$

Ans: $x \in (-\infty, 1) \cup (2, \infty)$

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6. Solve $|3x + 4| \geq 9$. [2]

$3x + 4 \geq 9$ or $3x + 4 \leq -9$

Solution set: $(-\infty, -\frac{13}{3}] \cup [\frac{5}{3}, \infty)$

Ans: $x \in (-\infty, -\frac{13}{3}] \cup [\frac{5}{3}, \infty)$

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7. Find $\lceil 6.3 \rceil - [-8.4]$. [1]

$\lceil 6.3 \rceil = 7,\ [-8.4] = -9$

$\lceil 6.3 \rceil - [-8.4] = 16$

Ans: 16

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8. Using $[x] + \left[x + \frac{1}{n}\right] + \cdots + \left[x + \frac{n-1}{n}\right] = [nx]$, evaluate $[0.46] + \left[0.46 + \frac{1}{5}\right] + \left[0.46 + \frac{2}{5}\right] + \left[0.46 + \frac{3}{5}\right] + \left[0.46 + \frac{4}{5}\right]$. [3]

Here $x = 0.46$, $n = 5$.

$\text{Sum} = [5 \times 0.46] = [2.3] = 2$

Check: the terms are 0, 0, 0, 1, 1.

Ans: 2

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9. Find $\log_4 8$. [1]

$\log_4 8 = \log_{2^2} 2^3$

$= \frac{3}{2} \log_2 2 = \frac{3}{2}$

Ans: $\frac{3}{2}$

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10. If $f(x) = 2x + 1$, find $(f \circ f)(x)$. [2]

$(f \circ f)(x) = f(2x + 1) = 2(2x + 1) + 1$

$= 4x + 3$

Ans: $(f \circ f)(x) = 4x + 3$

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11. If $f(x) = \dfrac{3x - 5}{2x - 3}$, show that $(f \circ f)(x) = x$ and hence find $(f \circ f \circ f)(-3)$. [5]

$(f \circ f)(x) = f\left(\frac{3x-5}{2x-3}\right) = \dfrac{3\left(\frac{3x-5}{2x-3}\right) - 5}{2\left(\frac{3x-5}{2x-3}\right) - 3}$

Multiply the numerator and the denominator by $2x - 3$:

$= \dfrac{(15 - 10x) + (9x - 15)}{(9 - 6x) + (6x - 10)} = \dfrac{-x}{-1}$

$= x$

So $f \circ f = I$, and $(f \circ f \circ f)(x) = f(x)$.

$(f \circ f \circ f)(-3) = f(-3) = \frac{-14}{-9} = \frac{14}{9}$

Ans: $(f \circ f)(x) = x$; $(f \circ f \circ f)(-3) = \frac{14}{9}$

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12. Write $f(x) = -2x^4 + 2x^3 + 4x^2 + 3x - 2$ as the sum of an even function and an odd function. [3]

$f(x) = \frac{1}{2}[f(x) + f(-x)] + \frac{1}{2}[f(x) - f(-x)]$

$f(-x) = -2x^4 - 2x^3 + 4x^2 - 3x - 2$

$\frac{1}{2}[f(x) + f(-x)] = -2x^4 + 4x^2 - 2$ (even)

$\frac{1}{2}[f(x) - f(-x)] = 2x^3 + 3x$ (odd)

$f(x) = \left(-2x^4 + 4x^2 - 2\right) + \left(2x^3 + 3x\right)$

Ans: Even part $-2x^4 + 4x^2 - 2$**, odd part** $2x^3 + 3x$

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13. If $f(x) = x - 3$ and $g(x) = x^2 + 6x + 5$, find the domain of $\dfrac{f}{g}$. [2]

$D_f = D_g = \mathbb{R}$, and $\frac{f}{g}$ needs $g(x) \neq 0$.

$(x + 1)(x + 5) = 0 \Rightarrow x = -5, -1$

Domain $= \mathbb{R} - \{-5, -1\}$

Ans: $\mathbb{R} - \{-5, -1\}$

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14. f satisfies $f(x+y) = f(x) + f(y)$ for all real x, y, and $f(2) = 4$. Find $f(3)$. [3]

$f(x+y) = f(x) + f(y) \Rightarrow f(x) = kx$.

$f(2) = 4 \Rightarrow 2k = 4 \Rightarrow k = 2$

$f(3) = 2 \times 3 = 6$

Ans: $f(3) = 6$

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15. How many onto functions are there from $A = \{1, 6, 7, 10, 11\}$ to $B = \{c, e\}$? [2]

$n(A) = 5,\ n(B) = 2$

$\text{Number of onto functions} = 2^n - 2 = 2^5 - 2 = 30$

Ans: 30

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16. Let $A = \mathbb{R} - \{5\}$ and $B = \mathbb{R} - \{6\}$. Show that $f : A \to B$, $f(x) = \frac{6x+2}{x-5}$ is a bijection and find f^{-1}. [5]

One-one: $f(x_1) = f(x_2) \Rightarrow (6x_1 + 2)(x_2 - 5) = (6x_2 + 2)(x_1 - 5)$

$\Rightarrow -32(x_1 - x_2) = 0 \Rightarrow x_1 = x_2$, so f is one-one.

Onto: for $y \in B$, $y = \frac{6x+2}{x-5} \Rightarrow y(x-5) = 6x + 2$

$\Rightarrow x = \frac{5y+2}{y-6}$, defined since $y \neq 6$, and $x \neq 5$ (else $-30 = 2$).

So $x \in A$ and $f(x) = y$: f is onto. Hence f is a bijection.

$f^{-1}(x) = \frac{5x+2}{x-6}$

Ans: $f^{-1}(x) = \frac{5x+2}{x-6}$

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17. Find the domain of $f(x) = \sqrt{x^2 - 9x + 20} + \dfrac{1}{x-1}$. [5]

We need $x^2 - 9x + 20 \geq 0$ and $x - 1 \neq 0$.

$x^2 - 9x + 20 = (x-5)(x-4) \geq 0$: the roots are $4 < 5$ and $a > 0$.

$\Rightarrow x \in (-\infty, 4] \cup [5, \infty)$

$x - 1 \neq 0 \Rightarrow x \neq 1$

Common part: domain $= (-\infty, 1) \cup (1, 4] \cup [5, \infty)$

Ans: $(-\infty, 1) \cup (1, 4] \cup [5, \infty)$

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18. Find the range of $f(x) = \dfrac{x^2 + 3}{x^2 + 2x + 5}$, $x \in \mathbb{R}$. [5]

The denominator $x^2 + 2x + 5$ has $\Delta = -16 < 0$, so it is never zero: the domain is $\mathbb{R}$.

$y = \frac{x^2+3}{x^2+2x+5} \Rightarrow y(x^2 + 2x + 5) = x^2 + 3$

$\Rightarrow (y-1)x^2 + (2y)x + (5y - 3) = 0$

For $y \neq 1$, x is real, so $\Delta = (2y)^2 - 4(y-1)(5y-3) \geq 0$

$\Rightarrow -16y^2 + 32y - 12 \geq 0, \Rightarrow \frac{1}{2} \leq y \leq \frac{3}{2}$

The value $y = 1$ is taken at $x = -1$, and 1 lies in this interval.

Range $= [\frac{1}{2}, \frac{3}{2}]$

Ans: $[\frac{1}{2}, \frac{3}{2}]$

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19. The point $(2, 7)$ lies on $y = f(x)$. Which point must lie on $y = 5f(x)$? [1]

The graph of $y = 5f(x)$ is obtained by vertical stretch: $(x, y) \to (x, 5y)$.

So $(2, 7) \to (2, 35)$.

Ans: $(2, 35)$

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20. Describe how to get the graph of $y = \sqrt{x-2} + 4$ from $y = \sqrt{x}$, and find its domain and range. [3]

$\sqrt{x-2}$: shift $y = \sqrt{x}$ right by 2.

Adding 4: shift up by 4.

$x - 2 \geq 0 \Rightarrow x \geq 2$

Domain $= [2, \infty)$; since $\sqrt{\ } \geq 0$, range $= [4, \infty)$.

Ans: Domain $[2, \infty)$**, range** $[4, \infty)$
