EXPRESSWALA ACADEMY

Maths 1A: Determinants

Question Paper, Set 08

Time: 1 hour 30 minutes Maximum marks: 50

Name: _____ Class: _____ Date: _____

1. Answer all 20 questions. The marks for each question are shown in brackets.
2. Show your working for the 2, 3 and 5 mark questions.
3. The key sheet and worked solutions are at the end of this paper.

1. Find the value of $\begin{vmatrix} 10 & -9 \\ 5 & 1 \end{vmatrix}$. [1 mark]

2. Find x if $\begin{vmatrix} x+2 & 4 \\ -5 & x-3 \end{vmatrix} = 44$. [2 marks]

3. For $A = \begin{bmatrix} -3 & -3 & 8 \\ 0 & 1 & -3 \\ 9 & 8 & -7 \end{bmatrix}$, find M_{22} and A_{22}. [2 marks]

4. Find the value of $\begin{vmatrix} -1 & 0 & 1 \\ 2 & 0 & 5 \\ 8 & -2 & 6 \end{vmatrix}$ by expanding along a suitable row or column. [3 marks]

5. If $\begin{vmatrix} a & b & c \\ p & q & r \\ x & y & z \end{vmatrix} = -3$, find $\begin{vmatrix} a & p & x \\ b & q & y \\ c & r & z \end{vmatrix}$. [1 mark]

6. Write the value of $\begin{vmatrix} -5 & 3 & -6 \\ 9 & 8 & -16 \\ -3 & -6 & 12 \end{vmatrix}$, giving a reason. [1 mark]

7. Simplify $\begin{vmatrix} 15 & -22 & -3 \\ 6 & -1 & 0 \\ 4 & 6 & 1 \end{vmatrix}$ by row and column operations and find its value. [3 marks]

8. For $A = \begin{bmatrix} 8 & -4 & 6 \\ -2 & 7 & -3 \\ 0 & 4 & -5 \end{bmatrix}$, with A_{ij} the cofactor of a_{ij}, find $a_{11}A_{12} + a_{21}A_{22} + a_{31}A_{32}$. [1 mark]

9. Find the values of x for which $\begin{vmatrix} x-6 & -x-6 & -x-6 \\ -x-6 & x-6 & -x-6 \\ -x-6 & -x-6 & x-6 \end{vmatrix} = 0$. [5 marks]

10. A is of order 3 and $|-2A| = 24$. Find $|A|$ and $|A^T|$. [2 marks]

11. Write the value of $\begin{vmatrix} 3 & 4 & 4 & 11 \\ 0 & 6 & 0 & 7 \\ 0 & 0 & -3 & 10 \\ 0 & 0 & 0 & 4 \end{vmatrix}$. [1 mark]

12. Evaluate $\begin{vmatrix} 2 & 7 & 6 \\ 7 & 6 & 2 \\ 6 & 2 & 7 \end{vmatrix}$ by a standard result. [2 marks]

13. Evaluate $\begin{vmatrix} 5 & -2 & -6 \\ -2 & 2 & 3 \\ -6 & 3 & 10 \end{vmatrix}$. [3 marks]

14. Show that $\begin{vmatrix} a^2+1 & ab & ac \\ ab & b^2+1 & bc \\ ac & bc & c^2+1 \end{vmatrix} = 1 + a^2 + b^2 + c^2$. [5 marks]

15. Express $\begin{vmatrix} (4-2)^2 & (4-3)^2 & (4-5)^2 \\ (5-2)^2 & (5-3)^2 & (5-5)^2 \\ (1-2)^2 & (1-3)^2 & (1-5)^2 \end{vmatrix}$ as the product of two determinants and hence find its value. [5 marks]

16. Differentiate $\Delta(x) = \begin{vmatrix} x^3 & x \\ 4 & -4 \end{vmatrix}$ by differentiating one row at a time. [2 marks]

17. If $\Delta(x) = \begin{vmatrix} x^2-1 & 2x & 4x-1 \\ 4-3x & 2x-2 & -3x-2 \\ 4 & -2 & 1 \end{vmatrix}$, find $\Delta'(x)$ and $\Delta'(-1)$. [5 marks]

18. Evaluate $\int_0^1 \begin{vmatrix} 2x+1 & e^x \\ -5 & -3 \end{vmatrix} \, dx$. [3 marks]

19. Is $\begin{bmatrix} 5 & 6 & 8 \\ 5 & 1 & -5 \\ 5 & 1 & 5 \end{bmatrix}$ singular or non-singular? [1 mark]

20. Find p if the matrix $\begin{bmatrix} -2 & p & 4 \\ -3 & 6 & -2 \\ 5 & -1 & 6 \end{bmatrix}$ is singular. [2 marks]

Key sheet

Q	Answer
1	55
2	$x = -5$ or 6
3	Minor -51, cofactor -51
4	-14
5	-3
6	0
7	-3
8	0
9	$x = -18,\ 0$
10	-3 and -3
11	-216
12	-315
13	15
14	$1 + a^2 + b^2 + c^2$
15	144
16	$\Delta'(x) = -12x^2 - 4$
17	$\Delta'(x) = -12x^2 - 64x - 18$, $\Delta'(-1) = 34$
18	$-11 + 5e$
19	Non-singular ($\lvert A\rvert = -250$)
20	$p = 22$

Solutions

1. Find the value of $\begin{vmatrix} 10 & -9 \\ 5 & 1 \end{vmatrix}$. [1]

$\begin{vmatrix} 10 & -9 \\ 5 & 1 \end{vmatrix} = 10 \cdot 1 - (-9) \cdot 5$

$= 10 - (-45) = 55$

Ans: 55

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2. Find x if $\begin{vmatrix} x+2 & 4 \\ -5 & x-3 \end{vmatrix} = 44$. [2]

$(x+2)(x-3) - 4 \cdot (-5) = 44$

$\Rightarrow x^2 - x - 30 = 0$

$\Rightarrow (x+5)(x-6) = 0$

$\therefore x = -5$ or $x = 6$

Ans: $x = -5$ or 6

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3. For $A = \begin{bmatrix} -3 & -3 & 8 \\ 0 & 1 & -3 \\ 9 & 8 & -7 \end{bmatrix}$, find M_{22} and A_{22}. [2]

Delete row 2 and column 2.

$M_{22} = \begin{vmatrix} -3 & 8 \\ 9 & -7 \end{vmatrix} = 21 - 72 = -51$

$A_{22} = (-1)^{2+2} M_{22} = -51$

Ans: Minor -51**, cofactor** -51

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4. Find the value of $\begin{vmatrix} -1 & 0 & 1 \\ 2 & 0 & 5 \\ 8 & -2 & 6 \end{vmatrix}$ by expanding along a suitable row or column. [3]

Expand along the second column.

$\Delta = 2 \begin{vmatrix} -1 & 1 \\ 2 & 5 \end{vmatrix}$

$= 2(-5 - 2)$

$= 2(-7) = -14$

Ans: -14

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5. If $\begin{vmatrix} a & b & c \\ p & q & r \\ x & y & z \end{vmatrix} = -3$, find $\begin{vmatrix} a & p & x \\ b & q & y \\ c & r & z \end{vmatrix}$. [1]

Rows and columns are interchanged: $|A^T| = |A|$.

$\therefore$ value $= 1 \times (-3) = -3$

Ans: -3

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6. Write the value of $\begin{vmatrix} -5 & 3 & -6 \\ 9 & 8 & -16 \\ -3 & -6 & 12 \end{vmatrix}$, giving a reason. [1]

$C_3 = -2C_2$: two columns are proportional.

$\therefore \Delta = 0$

Ans: 0

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7. Simplify $\begin{vmatrix} 15 & -22 & -3 \\ 6 & -1 & 0 \\ 4 & 6 & 1 \end{vmatrix}$ by row and column operations and find its value. [3]

Apply $R_1 \to R_1 - 3R_2$ (the value does not change):

$$\Delta = \begin{vmatrix} -3 & -19 & -3 \\ 6 & -1 & 0 \\ 4 & 6 & 1 \end{vmatrix}$$

Apply $C_2 \to C_2 - 5C_3$:

$$\Delta = \begin{vmatrix} -3 & -4 & -3 \\ 6 & -1 & 0 \\ 4 & 1 & 1 \end{vmatrix}$$

$$\Delta = -6\begin{vmatrix} -4 & -3 \\ 1 & 1 \end{vmatrix} - \begin{vmatrix} -3 & -3 \\ 4 & 1 \end{vmatrix}$$

$$= -6(-4-(-3)) - (-3-(-12))$$

$$= -6(-1) - (9) = -3$$

Ans: -3

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8. For $A = \begin{bmatrix} 8 & -4 & 6 \\ -2 & 7 & -3 \\ 0 & 4 & -5 \end{bmatrix}$, with A_{ij} the cofactor of a_{ij}, find $a_{11}A_{12} + a_{21}A_{22} + a_{31}A_{32}$. [1]

This is the elements of column 1 with the cofactors of column 2.

The sum of products of the elements of one line with the cofactors of a different line is 0.

$\therefore a_{11}A_{12} + a_{21}A_{22} + a_{31}A_{32} = 0$

Ans: 0

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9. Find the values of x for which $\begin{vmatrix} x-6 & -x-6 & -x-6 \\ -x-6 & x-6 & -x-6 \\ -x-6 & -x-6 & x-6 \end{vmatrix} = 0$. [5]

$C_1 \to C_1 + C_2 + C_3$: every entry of C_1 becomes $-x-18$.

$$\Delta = (-x-18)\begin{vmatrix} 1 & -x-6 & -x-6 \\ 1 & x-6 & -x-6 \\ 1 & -x-6 & x-6 \end{vmatrix}$$

$R_2 \to R_2 - R_1$, $R_3 \to R_3 - R_1$:

$$\Delta = (-x-18)\begin{vmatrix} 1 & -x-6 & -x-6 \\ 0 & 2x & 0 \\ 0 & 0 & 2x \end{vmatrix} = 4x^2(-x-18)$$

$\Delta = 0 \Rightarrow x = 0$ or $x = -18$

Ans: $x = -18,\ 0$

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10. A is of order 3 and $|-2A| = 24$. Find $|A|$ and $|A^T|$. [2]

$|-2A| = (-2)^3|A| = -8|A|$

$|A| = \frac{24}{-8} = -3$

$|A^T| = |A| = -3$

Ans: -3 **and** -3

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11. Write the value of $\begin{vmatrix} 3 & 4 & 4 & 11 \\ 0 & 6 & 0 & 7 \\ 0 & 0 & -3 & 10 \\ 0 & 0 & 0 & 4 \end{vmatrix}$. [1]

The matrix is triangular: the determinant is the product of the diagonal elements.

$3 \times 6 \times (-3) \times 4 = -216$

Ans: -216

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12. Evaluate $\begin{vmatrix} 2 & 7 & 6 \\ 7 & 6 & 2 \\ 6 & 2 & 7 \end{vmatrix}$ by a standard result. [2]

Use result 1, $-(a^3 + b^3 + c^3 - 3abc)$ with $a, b, c = 2, 7, 6$.

$-(8 + 343 + 216 - 252) = -315$

Ans: -315

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13. Evaluate $\begin{vmatrix} 5 & -2 & -6 \\ -2 & 2 & 3 \\ -6 & 3 & 10 \end{vmatrix}$. [3]

Result 10 with $a, b, c = -2, 1, 3$:

$1 + a^2 + b^2 + c^2 = 1 + 4 + 1 + 9 = 15$

Ans: 15

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14. Show that $\begin{vmatrix} a^2+1 & ab & ac \\ ab & b^2+1 & bc \\ ac & bc & c^2+1 \end{vmatrix} = 1 + a^2 + b^2 + c^2$. [5]

Multiply R_1, R_2, R_3 by a, b, c and divide by abc:

$\Delta = \frac{1}{abc}\begin{vmatrix} a+a^3 & a^2b & a^2c \\ ab^2 & b+b^3 & b^2c \\ ac^2 & bc^2 & c+c^3 \end{vmatrix}$

Take a, b, c out of C_1, C_2, C_3:

$\Delta = \begin{vmatrix} 1+a^2 & a^2 & a^2 \\ b^2 & 1+b^2 & b^2 \\ c^2 & c^2 & 1+c^2 \end{vmatrix}$

$R_1 \to R_1 + R_2 + R_3$, then $C_2 \to C_2 - C_1$, $C_3 \to C_3 - C_1$:

$\Delta = (1 + a^2 + b^2 + c^2)\begin{vmatrix} 1 & 0 & 0 \\ b^2 & 1 & 0 \\ c^2 & 0 & 1 \end{vmatrix} = 1 + a^2 + b^2 + c^2$

Ans: $1 + a^2 + b^2 + c^2$

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15. Express $\begin{vmatrix}(4-2)^2 & (4-3)^2 & (4-5)^2\\(5-2)^2 & (5-3)^2 & (5-5)^2\\(1-2)^2 & (1-3)^2 & (1-5)^2\end{vmatrix}$ as the product of two determinants and hence find its value. [5]

$(u-v)^2 = u^2\cdot 1 + 1\cdot v^2 - 2u\cdot v$

So, multiplying row by column,

$$\Delta = \begin{vmatrix}16 & 1 & -8\\25 & 1 & -10\\1 & 1 & -2\end{vmatrix}\begin{vmatrix}1 & 1 & 1\\4 & 9 & 25\\2 & 3 & 5\end{vmatrix}$$

$$\begin{vmatrix}16 & 1 & -8\\25 & 1 & -10\\1 & 1 & -2\end{vmatrix} = -24$$

$$\begin{vmatrix}1 & 1 & 1\\4 & 9 & 25\\2 & 3 & 5\end{vmatrix} = -6$$

$\therefore \Delta = (-24)\times(-6) = 144$

Ans: 144

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16. Differentiate $\Delta(x) = \begin{vmatrix}x^3 & x\\4 & -4\end{vmatrix}$ by differentiating one row at a time. [2]

$$\Delta'(x) = \begin{vmatrix}3x^2 & 1\\4 & -4\end{vmatrix} + \begin{vmatrix}x^3 & x\\0 & 0\end{vmatrix}$$

$= (-12x^2 - 4) + (0)$

$= -12x^2 - 4$

Ans: $\Delta'(x) = -12x^2 - 4$

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17. If $\Delta(x) = \begin{vmatrix}x^2-1 & 2x & 4x-1\\4-3x & 2x-2 & -3x-2\\4 & -2 & 1\end{vmatrix}$, find $\Delta'(x)$ and $\Delta'(-1)$. [5]

Differentiate one row at a time and add:

$$\Delta'(x) = \begin{vmatrix}2x & 2 & 4\\4-3x & 2x-2 & -3x-2\\4 & -2 & 1\end{vmatrix}$$

$$+\begin{vmatrix}x^2-1 & 2x & 4x-1\\-3 & 2 & -3\\4 & -2 & 1\end{vmatrix}$$

$$+\begin{vmatrix}x^2-1 & 2x & 4x-1\\4-3x & 2x-2 & -3x-2\\0 & 0 & 0\end{vmatrix}$$

The third determinant has a row of zeros, so it is 0.

$$\begin{vmatrix}2x & 2 & 4\\4-3x & 2x-2 & -3x-2\\4 & -2 & 1\end{vmatrix} = -8x^2 - 38x - 24$$

$$\begin{vmatrix}x^2-1 & 2x & 4x-1\\-3 & 2 & -3\\4 & -2 & 1\end{vmatrix} = -4x^2 - 26x + 6$$

$\therefore \Delta'(x) = -12x^2 - 64x - 18$

$\Delta'(-1) = 34$

Ans: $\Delta'(x) = -12x^2 - 64x - 18$, $\Delta'(-1) = 34$

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18. Evaluate $\int_0^1 \begin{vmatrix} 2x+1 & e^x \\ -5 & -3 \end{vmatrix} dx$. [3]

Only one row contains x, so integrate that row:

$\begin{vmatrix} \int_0^1 (2x+1)\,dx & \int_0^1 e^x\,dx \\ -5 & -3 \end{vmatrix} = \begin{vmatrix} 2 & -1+e \\ -5 & -3 \end{vmatrix}$

$= -11 + 5e$

Ans: $-11 + 5e$

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19. Is $\begin{bmatrix} 5 & 6 & 8 \\ 5 & 1 & -5 \\ 5 & 1 & 5 \end{bmatrix}$ singular or non-singular? [1]

Expand along the first row.

$|A| = 5\begin{vmatrix} 1 & -5 \\ 1 & 5 \end{vmatrix} - 6\begin{vmatrix} 5 & -5 \\ 5 & 5 \end{vmatrix} + 8\begin{vmatrix} 5 & 1 \\ 5 & 1 \end{vmatrix}$

$= 5(5-(-5)) - 6(25-(-25)) + 8(5-5)$

$= 5(10) - 6(50) + 8(0) = -250$

$|A| \neq 0$, so the matrix is non-singular.

Ans: Non-singular ($|A| = -250$)

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20. Find p if the matrix $\begin{bmatrix} -2 & p & 4 \\ -3 & 6 & -2 \\ 5 & -1 & 6 \end{bmatrix}$ is singular. [2]

Expand along any row; $|A|$ is linear in p:

$|A| = 8p - 176$

$|A| = 0 \Rightarrow p = 22$

Ans: $p = 22$
