EXPRESSWALA ACADEMY

Maths 1A: Determinants

Question Paper, Set 05

Time: 1 hour 30 minutes | Maximum marks: 50

Name: _____ Class: _____ Date: _____

1. Answer all 20 questions. The marks for each question are shown in brackets.
2. Show your working for the 2, 3 and 5 mark questions.
3. The key sheet and worked solutions are at the end of this paper.

1. Find the value of $\begin{vmatrix} -8 & 10 \\ 3 & 5 \end{vmatrix}$. [1 mark]

2. Solve $\begin{vmatrix} x-5 & 3 \\ -3 & x+3 \end{vmatrix} = -3$. [2 marks]

3. For $A = \begin{bmatrix} -4 & -2 & -5 \\ 0 & 6 & 4 \\ -2 & -6 & 9 \end{bmatrix}$, find M_{33} and A_{33}. [2 marks]

4. Find the value of $\begin{vmatrix} 4 & 5 & 0 \\ -5 & -5 & 8 \\ -3 & -3 & -2 \end{vmatrix}$ by expanding along a suitable row or column. [3 marks]

5. If $\begin{vmatrix} a & b & c \\ p & q & r \\ x & y & z \end{vmatrix} = -5$, find $\begin{vmatrix} p & q & r \\ x & y & z \\ a & b & c \end{vmatrix}$. [1 mark]

6. Without expanding, find the value of $\begin{vmatrix} -6 & -5 & -3 \\ -3 & -1 & 8 \\ -12 & -10 & -6 \end{vmatrix}$. [1 mark]

7. Using the properties of determinants, evaluate $\begin{vmatrix} 13 & 9 & 66 \\ 0 & -1 & -3 \\ 3 & 5 & 38 \end{vmatrix}$. [3 marks]

8. For $A = \begin{bmatrix} 1 & -3 & -4 \\ 2 & 6 & 0 \\ 1 & -2 & -4 \end{bmatrix}$, with A_{ij} the cofactor of a_{ij}, find $a_{11}A_{21} + a_{12}A_{22} + a_{13}A_{23}$. [1 mark]

9. Find the values of x for which $\begin{vmatrix} x & -3 & -3 \\ -3 & x & -3 \\ -3 & -3 & x \end{vmatrix} = 0$. [5 marks]

10. A is a square matrix of order 2 with $|A| = 4$. Find $|-3A|$ and the determinant of the cofactor matrix of A. [2 marks]

11. Find the determinant of $\begin{bmatrix} 1 & 11 & 8 & 4 \\ 0 & -3 & 1 & -4 \\ 0 & 0 & 1 & 12 \\ 0 & 0 & 0 & 6 \end{bmatrix}$. [1 mark]

12. Using a standard result, find $\begin{vmatrix} 6 & 8 & 2 \\ 8 & 2 & 6 \\ 2 & 6 & 8 \end{vmatrix}$. [2 marks]

13. Find the value of $\begin{vmatrix} 1 & 4 & -8 \\ 1 & 9 & -27 \\ 1 & 1 & -1 \end{vmatrix}$ using a standard result. [3 marks]

14. Show that $\begin{vmatrix} 1 & a^2 & a^3 \\ 1 & b^2 & b^3 \\ 1 & c^2 & c^3 \end{vmatrix} = (ab + bc + ca)(a - b)(b - c)(c - a)$. [5 marks]

15. By writing it as a product of two determinants, evaluate $\begin{vmatrix} (3+2)^2 & (3+3)^2 & (3+1)^2 \\ (1+2)^2 & (1+3)^2 & (1+1)^2 \\ (5+2)^2 & (5+3)^2 & (5+1)^2 \end{vmatrix}$. [5 marks]

16. If $\Delta(x) = \begin{vmatrix} 3x - 2 & x^2 \\ e^{2x} & 2x + 1 \end{vmatrix}$, find $\Delta'(x)$. [2 marks]

17. If $\Delta(x) = \begin{vmatrix} 1 & x & x^3 + x \\ -x & -x - 1 & 3 - 2x \\ 5 & -1 & -4 \end{vmatrix}$, find $\Delta'(x)$ and $\Delta'(-1)$. [5 marks]

18. Evaluate $\int_0^{\frac{\pi}{2}} \begin{vmatrix} \sin x & \cos x \\ -2 & 4 \end{vmatrix} dx$. [3 marks]

19. Is $\begin{bmatrix} 4 & 1 & -3 \\ -4 & -5 & -1 \\ 16 & 8 & -8 \end{bmatrix}$ singular or non-singular? [1 mark]

20. For what value of p is $\begin{bmatrix} p & -2 \\ -20 & -5 \end{bmatrix}$ singular? [2 marks]

Key sheet

Q	Answer
1	-70
2	$x = -1$ or 3
3	Minor -24, cofactor -24
4	-34
5	-5
6	0
7	-182
8	0
9	$x = -3,\ 6$
10	36 and 4
11	-18
12	-448
13	-22
14	$(ab+bc+ca)(a-b)(b-c)(c-a)$
15	64
16	$\Delta'(x) = -2x^2e^{2x} - 2xe^{2x} + 12x - 1$
17	$\Delta'(x) = 24x^3 + 15x^2 - 16x + 22$, $\Delta'(-1) = 29$
18	6
19	Singular ($\lvert A\rvert = 0$)
20	$p = -8$

Solutions

1. Find the value of $\begin{vmatrix} -8 & 10 \\ 3 & 5 \end{vmatrix}$. [1]

$\begin{vmatrix} -8 & 10 \\ 3 & 5 \end{vmatrix} = (-8)\cdot 5 - 10\cdot 3$

$= -40 - 30 = -70$

Ans: -70

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2. Solve $\begin{vmatrix} x-5 & 3 \\ -3 & x+3 \end{vmatrix} = -3$. [2]

$(x-5)(x+3) - 3\cdot(-3) = -3$

$\Rightarrow x^2 - 2x - 3 = 0$

$\Rightarrow (x+1)(x-3) = 0$

$\therefore x = -1$ or $x = 3$

Ans: $x = -1$ or 3

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3. For $A = \begin{bmatrix} -4 & -2 & -5 \\ 0 & 6 & 4 \\ -2 & -6 & 9 \end{bmatrix}$, find M_{33} and A_{33}. [2]

Delete row 3 and column 3.

$M_{33} = \begin{vmatrix} -4 & -2 \\ 0 & 6 \end{vmatrix} = -24 - 0 = -24$

$A_{33} = (-1)^{3+3} M_{33} = -24$

Ans: Minor -24**, cofactor** -24

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4. Find the value of $\begin{vmatrix} 4 & 5 & 0 \\ -5 & -5 & 8 \\ -3 & -3 & -2 \end{vmatrix}$ by expanding along a suitable row or column. [3]

Expand along the first row.

$\Delta = 4\begin{vmatrix} -5 & 8 \\ -3 & -2 \end{vmatrix} - 5\begin{vmatrix} -5 & 8 \\ -3 & -2 \end{vmatrix}$

$= 4(10 - (-24)) - 5(10 - (-24))$

$= 4(34) - 5(34) = -34$

Ans: -34

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5. If $\begin{vmatrix} a & b & c \\ p & q & r \\ x & y & z \end{vmatrix} = -5$, find $\begin{vmatrix} p & q & r \\ x & y & z \\ a & b & c \end{vmatrix}$. [1]

The rows are moved round once: two interchanges, so the sign changes twice.

$\therefore$ value $= 1 \times (-5) = -5$

Ans: -5

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6. Without expanding, find the value of $\begin{vmatrix} -6 & -5 & -3 \\ -3 & -1 & 8 \\ -12 & -10 & -6 \end{vmatrix}$. [1]

$R_3 = 2R_1$: two rows are proportional.

$\therefore \Delta = 0$

Ans: 0

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7. Using the properties of determinants, evaluate $\begin{vmatrix} 13 & 9 & 66 \\ 0 & -1 & -3 \\ 3 & 5 & 38 \end{vmatrix}$. [3]

Apply $R_1 \to R_1 - 2R_3$ (the value does not change):

$$\Delta = \begin{vmatrix} 7 & -1 & -10 \\ 0 & -1 & -3 \\ 3 & 5 & 38 \end{vmatrix}$$

Apply $C_3 \to C_3 - 8C_2$:

$$\Delta = \begin{vmatrix} 7 & -1 & -2 \\ 0 & -1 & 5 \\ 3 & 5 & -2 \end{vmatrix}$$

$$\Delta = -\begin{vmatrix} 7 & -2 \\ 3 & -2 \end{vmatrix} - 5\begin{vmatrix} 7 & -1 \\ 3 & 5 \end{vmatrix}$$

$$= -(-14 - (-6)) - 5(35 - (-3))$$

$$= -(-8) - 5(38) = -182$$

Ans: -182

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8. For $A = \begin{bmatrix} 1 & -3 & -4 \\ 2 & 6 & 0 \\ 1 & -2 & -4 \end{bmatrix}$, with A_{ij} the cofactor of a_{ij}, find $a_{11}A_{21} + a_{12}A_{22} + a_{13}A_{23}$. [1]

This is the elements of row 1 with the cofactors of row 2.

The sum of products of the elements of one line with the cofactors of a different line is 0.

$\therefore a_{11}A_{21} + a_{12}A_{22} + a_{13}A_{23} = 0$

Ans: 0

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9. Find the values of x for which $\begin{vmatrix} x & -3 & -3 \\ -3 & x & -3 \\ -3 & -3 & x \end{vmatrix} = 0$. [5]

At $x = -3$ all three rows are equal, so $(x+3)^2$ is a factor (factor theorem).

$C_1 \to C_1 + C_2 + C_3$:

$$\Delta = (x-6)\begin{vmatrix} 1 & -3 & -3 \\ 1 & x & -3 \\ 1 & -3 & x \end{vmatrix}$$

$R_2 \to R_2 - R_1$, $R_3 \to R_3 - R_1$:

$$\Delta = (x-6)\begin{vmatrix} 1 & -3 & -3 \\ 0 & x+3 & 0 \\ 0 & 0 & x+3 \end{vmatrix} = (x-6)(x+3)^2$$

$\Delta = 0 \Rightarrow x = -3$ or $x = 6$

Ans: $x = -3,\ 6$

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10. A is a square matrix of order 2 with $|A| = 4$. Find $|-3A|$ and the determinant of the cofactor matrix of A. [2]

$|-3A| = (-3)^2|A| = 9 \times 4 = 36$

$\det \text{ of cofactor matrix} = |A|^{2-1} = 4^1 = 4$

Ans: 36 **and** 4

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11. Find the determinant of $\begin{bmatrix} 1 & 11 & 8 & 4 \\ 0 & -3 & 1 & -4 \\ 0 & 0 & 1 & 12 \\ 0 & 0 & 0 & 6 \end{bmatrix}$. [1]

The matrix is triangular: the determinant is the product of the diagonal elements.

$1 \times (-3) \times 1 \times 6 = -18$

Ans: -18

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12. Using a standard result, find $\begin{vmatrix} 6 & 8 & 2 \\ 8 & 2 & 6 \\ 2 & 6 & 8 \end{vmatrix}$. [2]

Use result 1, $-(a^3 + b^3 + c^3 - 3abc)$ with $a, b, c = 6, 8, 2$.

$-(216 + 512 + 8 - 288) = -448$

Ans: -448

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13. Find the value of $\begin{vmatrix} 1 & 4 & -8 \\ 1 & 9 & -27 \\ 1 & 1 & -1 \end{vmatrix}$ using a standard result. [3]

Result 5 with $a, b, c = -2, -3, -1$:

$(ab + bc + ca)(a - b)(b - c)(c - a) = (11)(-2) = -22$

Ans: -22

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14. Show that $\begin{vmatrix} 1 & a^2 & a^3 \\ 1 & b^2 & b^3 \\ 1 & c^2 & c^3 \end{vmatrix} = (ab + bc + ca)(a - b)(b - c)(c - a)$. [5]

$R_1 \to R_1 - R_2$, $R_2 \to R_2 - R_3$, and take out $a - b$, $b - c$:

$$\Delta = (a-b)(b-c)\begin{vmatrix} 0 & a+b & a^2+ab+b^2 \\ 0 & b+c & b^2+bc+c^2 \\ 1 & c^2 & c^3 \end{vmatrix}$$

Expand along C_1:

$= (a-b)(b-c)\left[(a+b)(b^2+bc+c^2) - (b+c)(a^2+ab+b^2)\right]$

$= (a-b)(b-c)\left[ac^2 - a^2c + bc^2 - a^2b\right]$

$= (a-b)(b-c)(c-a)(ab+bc+ca)$

Ans: $(ab + bc + ca)(a - b)(b - c)(c - a)$

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15. By writing it as a product of two determinants, evaluate $\begin{vmatrix}(3+2)^2 & (3+3)^2 & (3+1)^2\\(1+2)^2 & (1+3)^2 & (1+1)^2\\(5+2)^2 & (5+3)^2 & (5+1)^2\end{vmatrix}$. [5]

$(u+v)^2 = u^2\cdot 1 + 1\cdot v^2 + 2u\cdot v$

So, multiplying row by column,

$$\Delta = \begin{vmatrix}9 & 1 & 6\\1 & 1 & 2\\25 & 1 & 10\end{vmatrix}\begin{vmatrix}1 & 1 & 1\\4 & 9 & 1\\2 & 3 & 1\end{vmatrix}$$

$$\begin{vmatrix}9 & 1 & 6\\1 & 1 & 2\\25 & 1 & 10\end{vmatrix} = -32$$

$$\begin{vmatrix}1 & 1 & 1\\4 & 9 & 1\\2 & 3 & 1\end{vmatrix} = -2$$

$\therefore \Delta = (-32)\times(-2) = 64$

Ans: 64

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16. If $\Delta(x) = \begin{vmatrix}3x-2 & x^2\\ e^{2x} & 2x+1\end{vmatrix}$, find $\Delta'(x)$. [2]

$$\Delta'(x) = \begin{vmatrix}3 & 2x\\ e^{2x} & 2x+1\end{vmatrix} + \begin{vmatrix}3x-2 & x^2\\ 2e^{2x} & 2\end{vmatrix}$$

$$= (-2xe^{2x} + 6x + 3) + (-2x^2e^{2x} + 6x - 4)$$

$$= -2x^2e^{2x} - 2xe^{2x} + 12x - 1$$

Ans: $\Delta'(x) = -2x^2e^{2x} - 2xe^{2x} + 12x - 1$

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17. If $\Delta(x) = \begin{vmatrix}1 & x & x^3+x\\ -x & -x-1 & 3-2x\\ 5 & -1 & -4\end{vmatrix}$, find $\Delta'(x)$ and $\Delta'(-1)$. [5]

Differentiate one row at a time and add:

$$\Delta'(x) = \begin{vmatrix}0 & 1 & 3x^2+1\\ -x & -x-1 & 3-2x\\ 5 & -1 & -4\end{vmatrix}$$

$$+\begin{vmatrix}1 & x & x^3+x\\ -1 & -1 & -2\\ 5 & -1 & -4\end{vmatrix}$$

$$+\begin{vmatrix}1 & x & x^3+x\\ -x & -x-1 & 3-2x\\ 0 & 0 & 0\end{vmatrix}$$

The third determinant has a row of zeros, so it is 0.

$$\begin{vmatrix}0 & 1 & 3x^2+1\\ -x & -x-1 & 3-2x\\ 5 & -1 & -4\end{vmatrix} = 18x^3 + 15x^2 - 8x + 20$$

$$\begin{vmatrix}1 & x & x^3+x\\ -1 & -1 & -2\\ 5 & -1 & -4\end{vmatrix} = 6x^3 - 8x + 2$$

$\therefore \Delta'(x) = 24x^3 + 15x^2 - 16x + 22$

$\Delta'(-1) = 29$

Ans: $\Delta'(x) = 24x^3 + 15x^2 - 16x + 22$, $\Delta'(-1) = 29$

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18. Evaluate $\int_0^{\frac{\pi}{2}} \begin{vmatrix} \sin x & \cos x \\ -2 & 4 \end{vmatrix} dx$. [3]

Only one row contains x, so integrate that row:

$\begin{vmatrix} \int_0^{\frac{\pi}{2}} \sin x\, dx & \int_0^{\frac{\pi}{2}} \cos x\, dx \\ -2 & 4 \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ -2 & 4 \end{vmatrix}$

$= 6$

Ans: 6

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19. Is $\begin{bmatrix} 4 & 1 & -3 \\ -4 & -5 & -1 \\ 16 & 8 & -8 \end{bmatrix}$ singular or non-singular? [1]

$R_3 = 3R_1 - R_2$, so $R_3 \to R_3 - (3R_1 - R_2)$ gives a row of zeros.

$|A| = 0$

So the matrix is singular.

Ans: Singular ($|A| = 0$)

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20. For what value of p is $\begin{bmatrix} p & -2 \\ -20 & -5 \end{bmatrix}$ singular? [2]

$\begin{vmatrix} p & -2 \\ -20 & -5 \end{vmatrix} = -5p - 40 = 0$

$\Rightarrow p = -8$

Ans: $p = -8$
