EXPRESSWALA ACADEMY

Maths 1A: Determinants

Question Paper, Set 04

Time: 1 hour 30 minutes Maximum marks: 50

Name: _____ Class: _____ Date: _____

1. Answer all 20 questions. The marks for each question are shown in brackets.
2. Show your working for the 2, 3 and 5 mark questions.
3. The key sheet and worked solutions are at the end of this paper.

1. Evaluate $\begin{vmatrix} -5 & -7 \\ 8 & 11 \end{vmatrix}$. [1 mark]

2. Find x if $\begin{vmatrix} x+2 & -4 \\ -1 & x+8 \end{vmatrix} = -9$. [2 marks]

3. Find the minor and the cofactor of the element 8 in $\begin{bmatrix} 8 & -5 & 9 \\ 3 & -4 & 0 \\ -2 & 4 & 3 \end{bmatrix}$. [2 marks]

4. Evaluate $\begin{vmatrix} -5 & 7 & 1 \\ 0 & -2 & 1 \\ 0 & -2 & -2 \end{vmatrix}$. [3 marks]

5. If $\begin{vmatrix} a & b & c \\ p & q & r \\ x & y & z \end{vmatrix} = -2$, find $\begin{vmatrix} -3a & b & c \\ -3p & q & r \\ -3x & y & z \end{vmatrix}$. [1 mark]

6. Write the value of $\begin{vmatrix} 3 & -6 & -24 \\ 5 & 8 & 32 \\ 8 & -1 & -4 \end{vmatrix}$, giving a reason. [1 mark]

7. Simplify $\begin{vmatrix} 7 & -3 & -20 \\ -3 & -2 & -6 \\ -9 & 5 & 34 \end{vmatrix}$ by row and column operations and find its value. [3 marks]

8. For $A = \begin{bmatrix} 1 & 1 & -2 \\ -3 & -3 & -5 \\ 3 & 2 & 0 \end{bmatrix}$, with A_{ij} the cofactor of a_{ij}, find $a_{12}A_{11} + a_{22}A_{21} + a_{32}A_{31}$. [1 mark]

9. Solve $\begin{vmatrix} x-5 & -x-5 & -x-5 \\ -x-5 & x-5 & -x-5 \\ -x-5 & -x-5 & x-5 \end{vmatrix} = 0$. [5 marks]

10. A is of order 3 and $|3A| = -108$. Find $|A|$ and $|A^T|$. [2 marks]

11. Write the value of $\begin{vmatrix} -4 & 0 & 0 \\ 12 & 2 & 0 \\ 8 & 11 & -1 \end{vmatrix}$. [1 mark]

12. Evaluate $\begin{vmatrix} 1 & 8 & 64 \\ 1 & 1 & 1 \\ 1 & -1 & 1 \end{vmatrix}$ by a standard result. [2 marks]

13. Evaluate $\begin{vmatrix} 36 & 49 & 64 \\ 49 & 64 & 81 \\ 64 & 81 & 100 \end{vmatrix}$. [3 marks]

14. Show that $\begin{vmatrix} 1 & a^2 & bc \\ 1 & b^2 & ac \\ 1 & c^2 & ab \end{vmatrix} = (a+b+c)(a-b)(b-c)(c-a)$. [5 marks]

15. Express $\begin{vmatrix} (3-5)^2 & (3-3)^2 & (3-4)^2 \\ (2-5)^2 & (2-3)^2 & (2-4)^2 \\ (1-5)^2 & (1-3)^2 & (1-4)^2 \end{vmatrix}$ as the product of two determinants and hence find its value. [5 marks]

16. Differentiate $\Delta(x) = \begin{vmatrix} 3x-2 & x^2+1 \\ \sin x & x^3 \end{vmatrix}$ by differentiating one row at a time. [2 marks]

17. If $\Delta(x) = \begin{vmatrix} 1 & x & 2x \\ 2-3x & 2x+3 & -2x-2 \\ 1 & 5 & -3 \end{vmatrix}$, find $\Delta'(x)$ and $\Delta'(-1)$. [5 marks]

18. Evaluate $\int_0^1 \begin{vmatrix} 4x^3 & x^2+x \\ 5 & 5 \end{vmatrix} dx$. [3 marks]

19. Is $\begin{bmatrix} -2 & 6 & 5 \\ -2 & -4 & 2 \\ 5 & 2 & 3 \end{bmatrix}$ singular or non-singular? [1 mark]

20. Find p if the matrix $\begin{bmatrix} -1 & 1 & 6 \\ 4 & p & 1 \\ 0 & 3 & 5 \end{bmatrix}$ is singular. [2 marks]

Key sheet

Q	Answer
1	1
2	$x = -7$ or -3
3	Minor -12, cofactor -12
4	-30
5	6
6	0
7	-74
8	0
9	$x = -15,\ 0$
10	-4 and -4
11	8
12	-126
13	-8
14	$(a+b+c)(a-b)(b-c)(c-a)$
15	-8
16	$\Delta'(x) = 12x^3 - x^2\cos x - 6x^2 - 2x\sin x - \cos x$
17	$\Delta'(x) = 22 - 90x$, $\Delta'(-1) = 112$
18	$\frac{5}{6}$
19	Non-singular ($\lvert A\rvert = 208$)
20	$p = 11$

Solutions

1. Evaluate $\begin{vmatrix} -5 & -7 \\ 8 & 11 \end{vmatrix}$. [1]

$$\begin{vmatrix} -5 & -7 \\ 8 & 11 \end{vmatrix} = (-5)\cdot 11 - (-7)\cdot 8$$

$$= -55 - (-56) = 1$$

Ans: 1

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2. Find x if $\begin{vmatrix} x+2 & -4 \\ -1 & x+8 \end{vmatrix} = -9$. [2]

$$(x+2)(x+8) - (-4)\cdot(-1) = -9$$

$$\Rightarrow x^2 + 10x + 21 = 0$$

$$\Rightarrow (x+7)(x+3) = 0$$

$$\therefore x = -7 \text{ or } x = -3$$

Ans: $x = -7$ or -3

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3. Find the minor and the cofactor of the element 8 in $\begin{bmatrix} 8 & -5 & 9 \\ 3 & -4 & 0 \\ -2 & 4 & 3 \end{bmatrix}$. [2]

Delete row 1 and column 1.

$$M_{11} = \begin{vmatrix} -4 & 0 \\ 4 & 3 \end{vmatrix} = -12 - 0 = -12$$

$$A_{11} = (-1)^{1+1} M_{11} = -12$$

Ans: Minor -12**, cofactor** -12

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4. Evaluate $\begin{vmatrix} -5 & 7 & 1 \\ 0 & -2 & 1 \\ 0 & -2 & -2 \end{vmatrix}$. [3]

Expand along the first column.

$$\Delta = -5 \begin{vmatrix} -2 & 1 \\ -2 & -2 \end{vmatrix}$$

$$= -5(4 - (-2))$$

$$= -5(6) = -30$$

Ans: -30

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5. If $\begin{vmatrix} a & b & c \\ p & q & r \\ x & y & z \end{vmatrix} = -2$, find $\begin{vmatrix} -3a & b & c \\ -3p & q & r \\ -3x & y & z \end{vmatrix}$. [1]

One column is multiplied by -3, so the value is multiplied by -3.

$$\therefore \text{value} = (-3) \times (-2) = 6$$

Ans: 6

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6. Write the value of $\begin{vmatrix} 3 & -6 & -24 \\ 5 & 8 & 32 \\ 8 & -1 & -4 \end{vmatrix}$, giving a reason. [1]

$C_3 = 4C_2$: two columns are proportional.

$\therefore \Delta = 0$

Ans: 0

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7. Simplify $\begin{vmatrix} 7 & -3 & -20 \\ -3 & -2 & -6 \\ -9 & 5 & 34 \end{vmatrix}$ by row and column operations and find its value. [3]

Apply $R_3 \to R_3 + 2R_1$ (the value does not change):

$$\Delta = \begin{vmatrix} 7 & -3 & -20 \\ -3 & -2 & -6 \\ 5 & -1 & -6 \end{vmatrix}$$

Apply $C_3 \to C_3 - 6C_2$:

$$\Delta = \begin{vmatrix} 7 & -3 & -2 \\ -3 & -2 & 6 \\ 5 & -1 & 0 \end{vmatrix}$$

$$\Delta = 5\begin{vmatrix} -3 & -2 \\ -2 & 6 \end{vmatrix} + \begin{vmatrix} 7 & -2 \\ -3 & 6 \end{vmatrix}$$

$$= 5(-18 - 4) + (42 - 6)$$

$$= 5(-22) + (36) = -74$$

Ans: -74

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8. For $A = \begin{bmatrix} 1 & 1 & -2 \\ -3 & -3 & -5 \\ 3 & 2 & 0 \end{bmatrix}$, with A_{ij} the cofactor of a_{ij}, find $a_{12}A_{11} + a_{22}A_{21} + a_{32}A_{31}$. [1]

This is the elements of column 2 with the cofactors of column 1.

The sum of products of the elements of one line with the cofactors of a different line is 0.

$\therefore a_{12}A_{11} + a_{22}A_{21} + a_{32}A_{31} = 0$

Ans: 0

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9. Solve $\begin{vmatrix} x-5 & -x-5 & -x-5 \\ -x-5 & x-5 & -x-5 \\ -x-5 & -x-5 & x-5 \end{vmatrix} = 0$. [5]

$C_1 \to C_1 + C_2 + C_3$: every entry of C_1 becomes $-x - 15$.

$$\Delta = (-x-15)\begin{vmatrix} 1 & -x-5 & -x-5 \\ 1 & x-5 & -x-5 \\ 1 & -x-5 & x-5 \end{vmatrix}$$

$R_2 \to R_2 - R_1$, $R_3 \to R_3 - R_1$:

$$\Delta = (-x-15)\begin{vmatrix} 1 & -x-5 & -x-5 \\ 0 & 2x & 0 \\ 0 & 0 & 2x \end{vmatrix} = 4x^2(-x-15)$$

$\Delta = 0 \Rightarrow x = 0$ or $x = -15$

Ans: $x = -15,\ 0$

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10. A is of order 3 and $|3A| = -108$. Find $|A|$ and $|A^T|$. [2]

$|3A| = 3^3|A| = 27|A|$

$|A| = \frac{-108}{27} = -4$

$|A^T| = |A| = -4$

Ans: -4 **and** -4

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11. Write the value of $\begin{vmatrix} -4 & 0 & 0 \\ 12 & 2 & 0 \\ 8 & 11 & -1 \end{vmatrix}$. [1]

The matrix is triangular: the determinant is the product of the diagonal elements.

$(-4) \times 2 \times (-1) = 8$

Ans: 8

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12. Evaluate $\begin{vmatrix} 1 & 8 & 64 \\ 1 & 1 & 1 \\ 1 & -1 & 1 \end{vmatrix}$ by a standard result. [2]

Use result 3, $(a-b)(b-c)(c-a)$ with $a, b, c = 8, 1, -1$.

$(8-1)(1-(-1))(-1-8) = -126$

Ans: -126

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13. Evaluate $\begin{vmatrix} 36 & 49 & 64 \\ 49 & 64 & 81 \\ 64 & 81 & 100 \end{vmatrix}$. [3]

The entries are squares of consecutive numbers, each row starting one later; this is result 11 with $n = 6$.

$\Delta = -8$

Ans: -8

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14. Show that $\begin{vmatrix} 1 & a^2 & bc \\ 1 & b^2 & ac \\ 1 & c^2 & ab \end{vmatrix} = (a+b+c)(a-b)(b-c)(c-a)$. [5]

$R_1 \to R_1 - R_2$, $R_2 \to R_2 - R_3$:

$$\Delta = \begin{vmatrix} 0 & (a-b)(a+b) & -c(a-b) \\ 0 & (b-c)(b+c) & -a(b-c) \\ 1 & c^2 & ab \end{vmatrix}$$

$$= (a-b)(b-c)\begin{vmatrix} 0 & a+b & -c \\ 0 & b+c & -a \\ 1 & c^2 & ab \end{vmatrix}$$

Expand along C_1:

$= (a-b)(b-c)\left[-a(a+b) + c(b+c)\right]$

$= (a-b)(b-c)\left[(c^2-a^2) + b(c-a)\right] = (a+b+c)(a-b)(b-c)(c-a)$

Ans: $(a+b+c)(a-b)(b-c)(c-a)$

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15. Express $\begin{vmatrix}(3-5)^2 & (3-3)^2 & (3-4)^2\\(2-5)^2 & (2-3)^2 & (2-4)^2\\(1-5)^2 & (1-3)^2 & (1-4)^2\end{vmatrix}$ as the product of two determinants and hence find its value. [5]

$(u-v)^2 = u^2\cdot 1 + 1\cdot v^2 - 2u\cdot v$

So, multiplying row by column,

$$\Delta = \begin{vmatrix}9 & 1 & -6\\4 & 1 & -4\\1 & 1 & -2\end{vmatrix}\begin{vmatrix}1 & 1 & 1\\25 & 9 & 16\\5 & 3 & 4\end{vmatrix}$$

$$\begin{vmatrix}9 & 1 & -6\\4 & 1 & -4\\1 & 1 & -2\end{vmatrix} = 4$$

$$\begin{vmatrix}1 & 1 & 1\\25 & 9 & 16\\5 & 3 & 4\end{vmatrix} = -2$$

$\therefore \Delta = 4\times(-2) = -8$

Ans: -8

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16. Differentiate $\Delta(x) = \begin{vmatrix}3x-2 & x^2+1\\ \sin x & x^3\end{vmatrix}$ by differentiating one row at a time. [2]

$$\Delta'(x) = \begin{vmatrix}3 & 2x\\ \sin x & x^3\end{vmatrix} + \begin{vmatrix}3x-2 & x^2+1\\ \cos x & 3x^2\end{vmatrix}$$

$$= (3x^3 - 2x\sin x) + (9x^3 - x^2\cos x - 6x^2 - \cos x)$$

$$= 12x^3 - x^2\cos x - 6x^2 - 2x\sin x - \cos x$$

Ans: $\Delta'(x) = 12x^3 - x^2\cos x - 6x^2 - 2x\sin x - \cos x$

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17. If $\Delta(x) = \begin{vmatrix}1 & x & 2x\\2-3x & 2x+3 & -2x-2\\1 & 5 & -3\end{vmatrix}$, find $\Delta'(x)$ and $\Delta'(-1)$. [5]

Differentiate one row at a time and add:

$$\Delta'(x) = \begin{vmatrix}0 & 1 & 2\\2-3x & 2x+3 & -2x-2\\1 & 5 & -3\end{vmatrix}$$

$$+\begin{vmatrix}1 & x & 2x\\-3 & 2 & -2\\1 & 5 & -3\end{vmatrix}$$

$$+\begin{vmatrix}1 & x & 2x\\2-3x & 2x+3 & -2x-2\\0 & 0 & 0\end{vmatrix}$$

The third determinant has a row of zeros, so it is 0.

$$\begin{vmatrix}0 & 1 & 2\\2-3x & 2x+3 & -2x-2\\1 & 5 & -3\end{vmatrix} = 18 - 45x$$

$$\begin{vmatrix}1 & x & 2x\\-3 & 2 & -2\\1 & 5 & -3\end{vmatrix} = 4 - 45x$$

$\therefore \Delta'(x) = 22 - 90x$

$\Delta'(-1) = 112$

Ans: $\Delta'(x) = 22 - 90x$, $\Delta'(-1) = 112$

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18. Evaluate $\int_0^1 \begin{vmatrix} 4x^3 & x^2+x \\ 5 & 5 \end{vmatrix} dx$. [3]

Only one row contains x, so integrate that row:

$\begin{vmatrix} \int_0^1 4x^3\,dx & \int_0^1 (x^2+x)\,dx \\ 5 & 5 \end{vmatrix} = \begin{vmatrix} 1 & \frac{5}{6} \\ 5 & 5 \end{vmatrix}$

$= \frac{5}{6}$

Ans: $\frac{5}{6}$

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19. Is $\begin{bmatrix} -2 & 6 & 5 \\ -2 & -4 & 2 \\ 5 & 2 & 3 \end{bmatrix}$ singular or non-singular? [1]

Expand along the first row.

$|A| = -2\begin{vmatrix} -4 & 2 \\ 2 & 3 \end{vmatrix} - 6\begin{vmatrix} -2 & 2 \\ 5 & 3 \end{vmatrix} + 5\begin{vmatrix} -2 & -4 \\ 5 & 2 \end{vmatrix}$

$= -2(-12-4) - 6(-6-10) + 5(-4-(-20))$

$= -2(-16) - 6(-16) + 5(16) = 208$

$|A| \neq 0$, so the matrix is non-singular.

Ans: Non-singular ($|A| = 208$)

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20. Find p if the matrix $\begin{bmatrix} -1 & 1 & 6 \\ 4 & p & 1 \\ 0 & 3 & 5 \end{bmatrix}$ is singular. [2]

Expand along any row; $|A|$ is linear in p:

$|A| = 55 - 5p$

$|A| = 0 \Rightarrow p = 11$

Ans: $p = 11$
