EXPRESSWALA ACADEMY

Maths 1B: Change of Axes

Question Paper, Set 09

Time: 1 hour 30 minutes | Maximum marks: 50

Name: _____ Class: _____ Date: _____

1. Answer all 20 questions. The marks for each question are shown in brackets.
2. Show your working for the 2, 3 and 5 mark questions.
3. The key sheet and worked solutions are at the end of this paper.

1. The origin is shifted to $(-3, 4)$. Find the new coordinates of the point $(-1, 6)$. [1 mark]

2. Find the transformed equation of $y^2 + 4x - 10y + 21 = 0$ when the origin is shifted to $(1, 5)$. [2 marks]

3. When the origin is shifted to $(-2, 1)$, the equation of a curve becomes $X^2 + 2Y^2 = 8$. Find its original equation. [2 marks]

4. Find the point to which the origin must be shifted to remove the first degree terms from $6x^2 - 2xy + 5y^2 + 14x - 12y - 9 = 0$. [3 marks]

5. Shift the origin to a suitable point so that the equation $-x^2 + 6xy - 2y^2 - 22x + 24y - 7 = 0$ has no first degree terms. Find the point and the transformed equation. [5 marks]

6. When the origin is shifted to $(-2, 4)$, the equation $-2x^2 + 3xy - 4y^2 - 3x + 5y + 3 = 0$ becomes $AX^2 + 2HXY + BY^2 + 2GX + 2FY + C = 0$. Find $2G$, $2F$ and C. [2 marks]

7. Show that $4x^2 + 2xy + 4y^2 - 6x + 6y + 6 = 0$ can be reduced to the form $aX^2 + 2hXY + bY^2 = 0$ by a translation of axes. Find the new origin and the reduced equation. [5 marks]

8. The axes are rotated through $150°$. Find the new coordinates of the point $(-5, 4)$. [1 mark]

9. The axes are rotated through an angle θ with $\cos\theta = \frac{4}{5}$, $\sin\theta = \frac{3}{5}$. A point has new coordinates $(-6, 7)$. Find its original coordinates. [2 marks]

10. Find the transformed equation of $x^2 + y^2 + x - 5y + 9 = 0$ when the axes are rotated through an angle θ with $\cos\theta = \frac{3}{5}$, $\sin\theta = \frac{4}{5}$. [2 marks]

11. After the axes are rotated through an angle θ with $\cos\theta = \frac{4}{5}$, $\sin\theta = \frac{3}{5}$, the equation of a curve becomes $X^2 - 3Y^2 = 2$. Find the original equation. [3 marks]

12. Find the angle through which the axes must be rotated to remove the xy term from $2x^2 + 2xy + 2y^2 = 3$. [1 mark]

13. Find the angle of rotation of the axes that removes the xy term from $6x^2 + 6xy - 2y^2 = 16$, and the transformed equation. [5 marks]

14. Find the angle through which the axes must be rotated so that the line $x + \sqrt{3}y + 8 = 0$ takes the form $Y = \text{constant}$. [1 mark]

15. Find the angle of rotation that reduces the line $6x + 8y - 11 = 0$ to the form $X = \text{constant}$, and find the new equation. [3 marks]

16. The axes are rotated through 30° and $3x^2 - 4xy - 2y^2 - x + 4y + 2 = 0$ becomes $AX^2 + 2HXY + BY^2 + 2GX + 2FY + C = 0$. Find $A + B$. [1 mark]

17. A rotation of the axes removes the xy term from $x^2 - 8xy + y^2 = 20$ and gives $AX^2 + BY^2 = 20$. Use the invariants to find A and B. [3 marks]

18. The origin is shifted to $(2, 4)$ and the axes are then rotated through 90°. Find the new coordinates of $(3, 3)$. [1 mark]

19. The origin is shifted to $(2, 3)$ and the axes are then rotated through an angle θ with $\cos\theta = \frac{4}{5}$, $\sin\theta = \frac{3}{5}$. A point has new coordinates $(-2, 3)$. Find its original coordinates. [2 marks]

20. Shift the origin to $(1, -3)$ and then rotate the axes through $\tan^{-1}\frac{1}{2}$. Find the transformed equation of $7x^2 + 12xy - 2y^2 + 22x - 24y - 31 = 0$. [5 marks]

Key sheet

Q	Answer
1	$(2, 2)$
2	$Y^2 + 4X = 0$
3	$x^2 + 2y^2 + 4x - 4y - 2 = 0$
4	$(-1, 1)$
5	$(-2, 3)$; $-X^2 + 6XY - 2Y^2 + 51 = 0$
6	$2G = 17,\ 2F = -33,\ C = -67$
7	New origin $(1, -1)$; $4X^2 + 2XY + 4Y^2 = 0$
8	$(2 + \frac{5\sqrt{3}}{2}, \frac{5}{2} - 2\sqrt{3})$
9	$(-9, 2)$
10	$5X^2 + 5Y^2 - 17X - 19Y + 45 = 0$
11	$11x^2 - 96xy + 39y^2 = -50$
12	$\theta = \frac{\pi}{4}$
13	$\theta = \tan^{-1}\frac{1}{3}$; $7X^2 - 3Y^2 = 16$
14	-30°
15	$\tan^{-1}\left(\frac{4}{3}\right)$; $X = \frac{11}{10}$
16	$A + B = 1$
17	-3 and 5 (in some order)
18	$(-1, -1)$
19	$(-\frac{7}{5}, \frac{21}{5})$
20	$10X^2 - 5Y^2 = -16$

Solutions

1. The origin is shifted to $(-3, 4)$. Find the new coordinates of the point $(-1, 6)$. [1]

$X = x - h,\ Y = y - k$

$X = -1 - (-3) = 2,\ Y = 6 - (4) = 2$

Ans: $(2, 2)$

---

2. Find the transformed equation of $y^2 + 4x - 10y + 21 = 0$ when the origin is shifted to $(1, 5)$. [2]

Put $x = X + 1,\ y = Y + 5$ in the equation.

Expanding and collecting terms: $Y^2 + 4X = 0$

Ans: $Y^2 + 4X = 0$

---

3. When the origin is shifted to $(-2, 1)$, the equation of a curve becomes $X^2 + 2Y^2 = 8$. Find its original equation. [2]

Replace X by $x + 2$ and Y by $y - 1$.

$x^2 + 2y^2 + 4x - 4y - 2 = 0$

Ans: $x^2 + 2y^2 + 4x - 4y - 2 = 0$

---

4. Find the point to which the origin must be shifted to remove the first degree terms from $6x^2 - 2xy + 5y^2 + 14x - 12y - 9 = 0$. [3]

$a = 6,\ h = -1,\ b = 5,\ g = 7,\ f = -6,\ c = -9$

Solve $ax + hy + g = 0$ and $hx + by + f = 0$: $6x - y + 7 = 0$, $-x + 5y - 6 = 0$

So $(x_1, y_1) = (-1, 1)$.

Ans: $(-1, 1)$

---

5. Shift the origin to a suitable point so that the equation $-x^2 + 6xy - 2y^2 - 22x + 24y - 7 = 0$ has no first degree terms. Find the point and the transformed equation. [5]

$a = -1,\ h = 3,\ b = -2,\ g = -11,\ f = 12,\ c = -7$

Solve $ax + hy + g = 0$ and $hx + by + f = 0$: $-x + 3y - 11 = 0$, $3x - 2y + 12 = 0$

So $(x_1, y_1) = (-2, 3)$.

Constant $= gx_1 + fy_1 + c = (-11)(-2) + (12)(3) + (-7) = 51$

Transformed equation: $-X^2 + 6XY - 2Y^2 + 51 = 0$

Ans: $(-2, 3)$; $-X^2 + 6XY - 2Y^2 + 51 = 0$

---

6. When the origin is shifted to $(-2, 4)$, the equation $-2x^2 + 3xy - 4y^2 - 3x + 5y + 3 = 0$ becomes $AX^2 + 2HXY + BY^2 + 2GX + 2FY + C = 0$. Find $2G$, $2F$ and C. [2]

$2G = \left(\frac{\partial S}{\partial x}\right)_{(-2,4)} = 2(-2)(-2) + (3)(4) + (-3) = 17$

$2F = \left(\frac{\partial S}{\partial y}\right)_{(-2,4)} = (3)(-2) + 2(-4)(4) + (5) = -33$

$C = S(-2, 4) = -67$

Ans: $2G = 17,\ 2F = -33,\ C = -67$

---

7. Show that $4x^2 + 2xy + 4y^2 - 6x + 6y + 6 = 0$ can be reduced to the form $aX^2 + 2hXY + bY^2 = 0$ by a translation of axes. Find the new origin and the reduced equation. [5]

$a = 4,\ h = 1,\ b = 4,\ g = -3,\ f = 3,\ c = 6$

$abc + 2fgh - af^2 - bg^2 - ch^2 = 96 - 18 - 36 - 36 - 6 = 0$

So the reduction is possible.

Solve $4x + y - 3 = 0$ and $x + 4y + 3 = 0$: new origin $(1, -1)$.

Constant $gx_1 + fy_1 + c = 0$, so the reduced equation is $4X^2 + 2XY + 4Y^2 = 0$

Ans: New origin $(1, -1)$; $4X^2 + 2XY + 4Y^2 = 0$

---

8. The axes are rotated through $150°$. Find the new coordinates of the point $(-5, 4)$. [1]

$X = x\cos\theta + y\sin\theta = 2 + \frac{5\sqrt{3}}{2},\ Y = -x\sin\theta + y\cos\theta = \frac{5}{2} - 2\sqrt{3}$

Ans: $(2 + \frac{5\sqrt{3}}{2}, \frac{5}{2} - 2\sqrt{3})$

---

9. The axes are rotated through an angle θ with $\cos\theta = \frac{4}{5}$, $\sin\theta = \frac{3}{5}$. A point has new coordinates $(-6, 7)$. Find its original coordinates. [2]

$x = X\cos\theta - Y\sin\theta = -9,\ y = X\sin\theta + Y\cos\theta = 2$

Ans: $(-9, 2)$

---

10. Find the transformed equation of $x^2 + y^2 + x - 5y + 9 = 0$ when the axes are rotated through an angle θ with $\cos\theta = \frac{3}{5}$, $\sin\theta = \frac{4}{5}$. [2]

Put $x = X\cos\theta - Y\sin\theta = \frac{3X}{5} - \frac{4Y}{5},\ y = X\sin\theta + Y\cos\theta = \frac{4X}{5} + \frac{3Y}{5}$.

Simplifying: $5X^2 + 5Y^2 - 17X - 19Y + 45 = 0$

Ans: $5X^2 + 5Y^2 - 17X - 19Y + 45 = 0$

---

11. After the axes are rotated through an angle θ with $\cos\theta = \frac{4}{5}$, $\sin\theta = \frac{3}{5}$, the equation of a curve becomes $X^2 - 3Y^2 = 2$. Find the original equation. [3]

Replace X by $x\cos\theta + y\sin\theta = \frac{4x}{5} + \frac{3y}{5}$ and Y by $-x\sin\theta + y\cos\theta = -\frac{3x}{5} + \frac{4y}{5}$.

Simplifying: $11x^2 - 96xy + 39y^2 = -50$

Ans: $11x^2 - 96xy + 39y^2 = -50$

---

12. Find the angle through which the axes must be rotated to remove the xy term from $2x^2 + 2xy + 2y^2 = 3$. [1]

Here $a = b = 2$, so $\theta = \frac{\pi}{4}$.

Ans: $\theta = \frac{\pi}{4}$

---

13. Find the angle of rotation of the axes that removes the xy term from $6x^2 + 6xy - 2y^2 = 16$, and the transformed equation. [5]

$\tan 2\theta = \frac{2h}{a-b} = \frac{3}{4}$, so $\cos\theta = \frac{3\sqrt{10}}{10},\ \sin\theta = \frac{\sqrt{10}}{10}$.

Put $x = \frac{3\sqrt{10}X}{10} - \frac{\sqrt{10}Y}{10},\ y = \frac{\sqrt{10}X}{10} + \frac{3\sqrt{10}Y}{10}$.

The XY terms cancel and the equation becomes $7X^2 - 3Y^2 = 16$

Check: $A + B = 4 = a + b$.

Ans: $\theta = \tan^{-1}\frac{1}{3}$; $7X^2 - 3Y^2 = 16$

---

14. Find the angle through which the axes must be rotated so that the line $x + \sqrt{3}y + 8 = 0$ takes the form $Y = \text{constant}$. [1]

Angle $= \tan^{-1}\left(-\frac{a}{b}\right) = -30^\circ$.

Ans: -30°

---

15. Find the angle of rotation that reduces the line $6x + 8y - 11 = 0$ to the form $X = \text{constant}$, and find the new equation. [3]

Angle $= \tan^{-1}\left(\frac{b}{a}\right) = \tan^{-1}\left(\frac{4}{3}\right)$.

Here $\cos\theta = \frac{3}{5},\ \sin\theta = \frac{4}{5}$; put $x = \frac{3X}{5} - \frac{4Y}{5},\ y = \frac{4X}{5} + \frac{3Y}{5}$.

The line becomes $10X - 11 = 0$, that is $X = \frac{11}{10}$

Ans: $\tan^{-1}\left(\frac{4}{3}\right)$; $X = \frac{11}{10}$

---

16. The axes are rotated through 30° and $3x^2 - 4xy - 2y^2 - x + 4y + 2 = 0$ becomes $AX^2 + 2HXY + BY^2 + 2GX + 2FY + C = 0$. Find $A + B$. [1]

$A + B = a + b$ (an invariant under rotation).

$= 1$

Ans: $A + B = 1$

---

17. A rotation of the axes removes the xy term from $x^2 - 8xy + y^2 = 20$ and gives $AX^2 + BY^2 = 20$. Use the invariants to find A and B. [3]

$A + B = a + b = 2$

$H^2 - AB = h^2 - ab$ with $H = 0$: $AB = ab - h^2 = -15$

So A and B are the roots of $t^2 - (2)t + (-15) = 0$: -3 and 5 (in some order).

Ans: -3 **and** 5 **(in some order)**

---

18. The origin is shifted to $(2, 4)$ and the axes are then rotated through 90°. Find the new coordinates of $(3, 3)$. [1]

$x - h = 1,\ y - k = -1$

$X = (x - h)\cos\theta + (y - k)\sin\theta = -1$

$Y = -(x - h)\sin\theta + (y - k)\cos\theta = -1$

Ans: $(-1, -1)$

---

19. The origin is shifted to $(2, 3)$ and the axes are then rotated through an angle θ with $\cos\theta = \frac{4}{5}$, $\sin\theta = \frac{3}{5}$. A point has new coordinates $(-2, 3)$. Find its original coordinates. [2]

$x - h = X\cos\theta - Y\sin\theta = -\frac{17}{5}$

$y - k = X\sin\theta + Y\cos\theta = \frac{6}{5}$

So $(x, y) = (-\frac{7}{5}, \frac{21}{5})$.

Ans: $(-\frac{7}{5}, \frac{21}{5})$

---

20. Shift the origin to $(1, -3)$ and then rotate the axes through $\tan^{-1}\frac{1}{2}$. Find the transformed equation of $7x^2 + 12xy - 2y^2 + 22x - 24y - 31 = 0$. [5]

Shift the origin: $x - h = x - 1,\ y - k = y + 3$. The equation is $7(x-1)^2 + 12(x-1)(y+3) - 2(y+3)^2 = -16$.

Rotate: $\cos\theta = \frac{2\sqrt{5}}{5},\ \sin\theta = \frac{\sqrt{5}}{5}$; put $x - h = \frac{2\sqrt{5}X}{5} - \frac{\sqrt{5}Y}{5},\ y - k = \frac{\sqrt{5}X}{5} + \frac{2\sqrt{5}Y}{5}$.

The XY terms cancel: $10X^2 - 5Y^2 = -16$

Ans: $10X^2 - 5Y^2 = -16$