EXPRESSWALA ACADEMY

Maths 1B: Change of Axes

Question Paper, Set 08

Time: 1 hour 30 minutes | Maximum marks: 50

Name: _____ Class: _____ Date: _____

1. Answer all 20 questions. The marks for each question are shown in brackets.
2. Show your working for the 2, 3 and 5 mark questions.
3. The key sheet and worked solutions are at the end of this paper.

1. The origin is shifted to $(6, 1)$. A point has new coordinates $(7, 9)$. Find its original coordinates. [1 mark]

2. Find the transformed equation of $3x^2 + 4xy + 4y^2 + 3x + 6y + 5 = 0$ when the origin is shifted to $(-1, 3)$. [2 marks]

3. When the origin is shifted to $(2, 2)$, the equation of a curve becomes $X^2 + Y^2 = 10$. Find its original equation. [2 marks]

4. Find the point to which the origin must be shifted to remove the first degree terms from $-2x^2 + 4y^2 - 8x - 8y + 1 = 0$. [3 marks]

5. Shift the origin to a suitable point so that the equation $x^2 + 4xy + 5y^2 + 2x + 2y - 3 = 0$ has no first degree terms. Find the point and the transformed equation. [5 marks]

6. When the origin is shifted to $(3, 1)$, the equation $-4x^2 + xy + 4y^2 + x + 3y + 3 = 0$ becomes $AX^2 + 2HXY + BY^2 + 2GX + 2FY + C = 0$. Find $2G$, $2F$ and C. [2 marks]

7. Show that $4x^2 + 4xy + 5y^2 - 20x - 18y + 29 = 0$ can be reduced to the form $aX^2 + 2hXY + bY^2 = 0$ by a translation of axes. Find the new origin and the reduced equation. [5 marks]

8. The axes are rotated through 45°. Find the new coordinates of the point $(7, -5)$. [1 mark]

9. The axes are rotated through 60° in the clockwise direction. Find the new coordinates of the point $(-1, 6)$. [2 marks]

10. Find the transformed equation of $3x - 2y + 1 = 0$ when the axes are rotated through 60°. [2 marks]

11. After the axes are rotated through 90°, the equation of a curve becomes $XY = 6$. Find the original equation. [3 marks]

12. Find the angle through which the axes must be rotated to remove the xy term from $4x^2 + 6\sqrt{3}xy - 2y^2 = 3$. [1 mark]

13. Transform $4x^2 + 4xy + 4y^2 = 3$ by rotating the axes so that the xy term disappears. [5 marks]

14. Find the angle through which the axes must be rotated so that the line $-4x + 3y - 9 = 0$ takes the form $X = \text{constant}$. [1 mark]

15. Find the angle of rotation that reduces the line $8x - 6y + 5 = 0$ to the form $Y = \text{constant}$, and find the new equation. [3 marks]

16. The axes are rotated through 30° and $-4x^2 + 3xy + 3y^2 - 8x + 7y - 6 = 0$ becomes $AX^2 + 2HXY + BY^2 + 2GX + 2FY + C = 0$. Find $(A - B)^2 + 4H^2$. [1 mark]

17. A rotation of the axes removes the xy term from $-4x^2 - 6xy + 4y^2 = 8$ and gives $AX^2 + BY^2 = 8$. Use the invariants to find A and B. [3 marks]

18. The origin is shifted to $(1, 2)$ and the axes are then rotated through 180°. Find the new coordinates of $(6, 5)$. [1 mark]

19. The origin is shifted to $(-3, 5)$ and the axes are then rotated through an angle θ with $\cos\theta = \frac{4}{5}$, $\sin\theta = \frac{3}{5}$. A point has new coordinates $(6, 7)$. Find its original coordinates. [2 marks]

20. Shift the origin to $(-3, -3)$ and then rotate the axes through 45°. Find the transformed equation of $-2x^2 - 6xy - 2y^2 - 30x - 30y - 96 = 0$. [5 marks]

Key sheet

Q	Answer
1	$(13, 10)$
2	$3X^2 + 4XY + 4Y^2 + 9X + 26Y + 47 = 0$
3	$x^2 + y^2 - 4x - 4y - 2 = 0$
4	$(-2, 1)$
5	$(-3, 1)$; $X^2 + 4XY + 5Y^2 - 5 = 0$
6	$2G = -22,\ 2F = 14,\ C = -20$
7	New origin $(2, 1)$; $4X^2 + 4XY + 5Y^2 = 0$
8	$(\sqrt{2}, -6\sqrt{2})$
9	$(-3\sqrt{3} - \frac{1}{2}, 3 - \frac{\sqrt{3}}{2})$
10	$-3 + 2\sqrt{3}X + 2 + 3\sqrt{3}Y - 2 = 0$
11	$xy = -6$
12	$\theta = \frac{\pi}{6}$
13	$\theta = \frac{\pi}{4}$; $6X^2 + 2Y^2 = 3$
14	$\tan^{-1}\left(-\frac{3}{4}\right)$
15	$\tan^{-1}\left(\frac{4}{3}\right)$; $Y = \frac{1}{2}$
16	$(A - B)^2 + 4H^2 = 58$
17	-5 and 5 (in some order)
18	$(-5, -3)$
19	$(-\frac{12}{5}, \frac{71}{5})$
20	$-5X^2 + Y^2 = 6$

Solutions

1. The origin is shifted to $(6, 1)$. A point has new coordinates $(7, 9)$. Find its original coordinates. [1]

$x = X + h,\ y = Y + k$

$x = 7 + (6) = 13,\ y = 9 + (1) = 10$

Ans: $(13, 10)$

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2. Find the transformed equation of $3x^2 + 4xy + 4y^2 + 3x + 6y + 5 = 0$ when the origin is shifted to $(-1, 3)$. [2]

Put $x = X - 1,\ y = Y + 3$ in the equation.

Expanding and collecting terms: $3X^2 + 4XY + 4Y^2 + 9X + 26Y + 47 = 0$

Ans: $3X^2 + 4XY + 4Y^2 + 9X + 26Y + 47 = 0$

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3. When the origin is shifted to $(2, 2)$, the equation of a curve becomes $X^2 + Y^2 = 10$. Find its original equation. [2]

Replace X by $x - 2$ and Y by $y - 2$.

$x^2 + y^2 - 4x - 4y - 2 = 0$

Ans: $x^2 + y^2 - 4x - 4y - 2 = 0$

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4. Find the point to which the origin must be shifted to remove the first degree terms from $-2x^2 + 4y^2 - 8x - 8y + 1 = 0$. [3]

$a = -2,\ h = 0,\ b = 4,\ g = -4,\ f = -4,\ c = 1$

Solve $ax + hy + g = 0$ and $hx + by + f = 0$: $-2x - 4 = 0$, $4y - 4 = 0$

So $(x_1, y_1) = (-2, 1)$.

Ans: $(-2, 1)$

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5. Shift the origin to a suitable point so that the equation $x^2 + 4xy + 5y^2 + 2x + 2y - 3 = 0$ has no first degree terms. Find the point and the transformed equation. [5]

$a = 1,\ h = 2,\ b = 5,\ g = 1,\ f = 1,\ c = -3$

Solve $ax + hy + g = 0$ and $hx + by + f = 0$: $x + 2y + 1 = 0$, $2x + 5y + 1 = 0$

So $(x_1, y_1) = (-3, 1)$.

Constant $= gx_1 + fy_1 + c = (1)(-3) + (1)(1) + (-3) = -5$

Transformed equation: $X^2 + 4XY + 5Y^2 - 5 = 0$

Ans: $(-3, 1)$; $X^2 + 4XY + 5Y^2 - 5 = 0$

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6. When the origin is shifted to $(3, 1)$, the equation $-4x^2 + xy + 4y^2 + x + 3y + 3 = 0$ becomes $AX^2 + 2HXY + BY^2 + 2GX + 2FY + C = 0$. Find $2G$, $2F$ and C. [2]

$2G = \left(\frac{\partial S}{\partial x}\right)_{(3,1)} = 2(-4)(3) + (1)(1) + (1) = -22$

$2F = \left(\frac{\partial S}{\partial y}\right)_{(3,1)} = (1)(3) + 2(4)(1) + (3) = 14$

$C = S(3, 1) = -20$

Ans: $2G = -22,\ 2F = 14,\ C = -20$

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7. Show that $4x^2 + 4xy + 5y^2 - 20x - 18y + 29 = 0$ can be reduced to the form $aX^2 + 2hXY + bY^2 = 0$ by a translation of axes. Find the new origin and the reduced equation. [5]

$a = 4,\ h = 2,\ b = 5,\ g = -10,\ f = -9,\ c = 29$

$abc + 2fgh - af^2 - bg^2 - ch^2 = 580 + 360 - 324 - 500 - 116 = 0$

So the reduction is possible.

Solve $4x + 2y - 10 = 0$ and $2x + 5y - 9 = 0$: new origin $(2, 1)$.

Constant $gx_1 + fy_1 + c = 0$, so the reduced equation is $4X^2 + 4XY + 5Y^2 = 0$

Ans: New origin $(2, 1)$; $4X^2 + 4XY + 5Y^2 = 0$

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8. The axes are rotated through $45°$. Find the new coordinates of the point $(7, -5)$. [1]

$X = x\cos\theta + y\sin\theta = \sqrt{2},\ Y = -x\sin\theta + y\cos\theta = -6\sqrt{2}$

Ans: $(\sqrt{2}, -6\sqrt{2})$

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9. The axes are rotated through $60°$ in the clockwise direction. Find the new coordinates of the point $(-1, 6)$. [2]

Clockwise: use $\theta = -60°$.

$X = x\cos\theta + y\sin\theta = -3\sqrt{3} - \frac{1}{2},\ Y = -x\sin\theta + y\cos\theta = 3 - \frac{\sqrt{3}}{2}$

Ans: $(-3\sqrt{3} - \frac{1}{2}, 3 - \frac{\sqrt{3}}{2})$

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10. Find the transformed equation of $3x - 2y + 1 = 0$ when the axes are rotated through $60°$. [2]

Put $x = X\cos\theta - Y\sin\theta = \frac{X}{2} - \frac{\sqrt{3}Y}{2},\ y = X\sin\theta + Y\cos\theta = \frac{\sqrt{3}X}{2} + \frac{Y}{2}$.

Simplifying: $-3 + 2\sqrt{3}X + 2 + 3\sqrt{3}Y - 2 = 0$

Ans: $-3 + 2\sqrt{3}X + 2 + 3\sqrt{3}Y - 2 = 0$

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11. After the axes are rotated through $90°$, the equation of a curve becomes $XY = 6$. Find the original equation. [3]

Replace X by $x\cos\theta + y\sin\theta = y$ and Y by $-x\sin\theta + y\cos\theta = -x$.

Simplifying: $xy = -6$

Ans: $xy = -6$

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12. Find the angle through which the axes must be rotated to remove the xy term from $4x^2 + 6\sqrt{3}xy - 2y^2 = 3$. [1]

$\tan 2\theta = \frac{2h}{a-b} = \frac{6\sqrt{3}}{4-(-2)} = \sqrt{3}$

$\theta = \frac{\pi}{6} = \frac{\pi}{6}$

Ans: $\theta = \frac{\pi}{6}$

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13. Transform $4x^2 + 4xy + 4y^2 = 3$ by rotating the axes so that the xy term disappears. [5]

Here $a = b$, so $\theta = \frac{\pi}{4}$.

Put $x = \frac{\sqrt{2}X}{2} - \frac{\sqrt{2}Y}{2},\ y = \frac{\sqrt{2}X}{2} + \frac{\sqrt{2}Y}{2}$.

The XY terms cancel and the equation becomes $6X^2 + 2Y^2 = 3$

Check: $A + B = 8 = a + b$.

Ans: $\theta = \frac{\pi}{4}$; $6X^2 + 2Y^2 = 3$

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14. Find the angle through which the axes must be rotated so that the line $-4x + 3y - 9 = 0$ takes the form $X = \text{constant}$. [1]

Angle $= \tan^{-1}\left(\frac{b}{a}\right) = \tan^{-1}\left(-\frac{3}{4}\right)$.

Ans: $\tan^{-1}\left(-\frac{3}{4}\right)$

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15. Find the angle of rotation that reduces the line $8x - 6y + 5 = 0$ to the form $Y = \text{constant}$, and find the new equation. [3]

Angle $= \tan^{-1}\left(-\frac{a}{b}\right) = \tan^{-1}\left(\frac{4}{3}\right)$.

Here $\cos\theta = \frac{3}{5},\ \sin\theta = \frac{4}{5}$; put $x = \frac{3X}{5} - \frac{4Y}{5},\ y = \frac{4X}{5} + \frac{3Y}{5}$.

The line becomes $-10Y + 5 = 0$, that is $Y = \frac{1}{2}$

Ans: $\tan^{-1}\left(\frac{4}{3}\right)$; $Y = \frac{1}{2}$

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16. The axes are rotated through 30° and $-4x^2 + 3xy + 3y^2 - 8x + 7y - 6 = 0$ becomes $AX^2 + 2HXY + BY^2 + 2GX + 2FY + C = 0$. Find $(A - B)^2 + 4H^2$. [1]

$(A - B)^2 + 4H^2 = (a - b)^2 + 4h^2$ (an invariant under rotation).

$= 58$

Ans: $(A - B)^2 + 4H^2 = 58$

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17. A rotation of the axes removes the xy term from $-4x^2 - 6xy + 4y^2 = 8$ and gives $AX^2 + BY^2 = 8$. Use the invariants to find A and B. [3]

$A + B = a + b = 0$

$H^2 - AB = h^2 - ab$ with $H = 0$: $AB = ab - h^2 = -25$

So A and B are the roots of $t^2 - (0)t + (-25) = 0$: -5 and 5 (in some order).

Ans: -5 **and** 5 **(in some order)**

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18. The origin is shifted to $(1, 2)$ and the axes are then rotated through 180°. Find the new coordinates of $(6, 5)$. [1]

$x - h = 5,\ y - k = 3$

$X = (x - h)\cos\theta + (y - k)\sin\theta = -5$

$Y = -(x - h)\sin\theta + (y - k)\cos\theta = -3$

Ans: $(-5, -3)$

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19. The origin is shifted to $(-3, 5)$ and the axes are then rotated through an angle θ with $\cos\theta = \frac{4}{5}$, $\sin\theta = \frac{3}{5}$. A point has new coordinates $(6, 7)$. Find its original coordinates. [2]

$x - h = X\cos\theta - Y\sin\theta = \frac{3}{5}$

$y - k = X\sin\theta + Y\cos\theta = \frac{46}{5}$

So $(x, y) = (-\frac{12}{5}, \frac{71}{5})$.

Ans: $(-\frac{12}{5}, \frac{71}{5})$

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20. Shift the origin to $(-3, -3)$ and then rotate the axes through 45°. Find the transformed equation of $-2x^2 - 6xy - 2y^2 - 30x - 30y - 96 = 0$. [5]

Shift the origin: $x - h = x + 3,\ y - k = y + 3$. The equation is $-2(x+3)^2 - 6(x+3)(y+3) - 2(y+3)^2 = 6$.

Rotate: $\cos\theta = \frac{\sqrt{2}}{2},\ \sin\theta = \frac{\sqrt{2}}{2}$; put $x - h = \frac{\sqrt{2}X}{2} - \frac{\sqrt{2}Y}{2},\ y - k = \frac{\sqrt{2}X}{2} + \frac{\sqrt{2}Y}{2}$.

The XY terms cancel: $-5X^2 + Y^2 = 6$

Ans: $-5X^2 + Y^2 = 6$
