EXPRESSWALA ACADEMY

Maths 1B: Change of Axes

Question Paper, Set 04

Time: 1 hour 30 minutes | Maximum marks: 50

Name: _____ Class: _____ Date: _____

1. Answer all 20 questions. The marks for each question are shown in brackets.
2. Show your working for the 2, 3 and 5 mark questions.
3. The key sheet and worked solutions are at the end of this paper.

1. The origin is shifted to $(6, 6)$. A point has new coordinates $(-8, 6)$. Find its original coordinates. [1 mark]

2. Find the transformed equation of $3x^2 + xy + 2y^2 + 5x + 8y - 4 = 0$ when the origin is shifted to $(-4, -4)$. [2 marks]

3. When the origin is shifted to $(4, -1)$, the equation of a curve becomes $4X^2 - 5Y^2 = 20$. Find its original equation. [2 marks]

4. Find the point to which the origin must be shifted to remove the first degree terms from $x^2 + 4xy + 2y^2 - 14x - 8y - 2 = 0$. [3 marks]

5. Shift the origin to a suitable point so that the equation $-x^2 + 6xy - y^2 + 8x - 8y + 5 = 0$ has no first degree terms. Find the point and the transformed equation. [5 marks]

6. When the origin is shifted to $(-2, -3)$, the equation $2x^2 + 3y^2 - 5x - 2y = 0$ becomes $AX^2 + 2HXY + BY^2 + 2GX + 2FY + C = 0$. Find $2G$, $2F$ and C. [2 marks]

7. Show that $x^2 - y^2 - 8x - 6y + 7 = 0$ can be reduced to the form $aX^2 + 2hXY + bY^2 = 0$ by a translation of axes. Find the new origin and the reduced equation. [5 marks]

8. The axes are rotated through 180°. Find the new coordinates of the point $(1, 3)$. [1 mark]

9. The axes are rotated through 45° in the clockwise direction. Find the new coordinates of the point $(-4, -3)$. [2 marks]

10. Find the transformed equation of $xy = -5$ when the axes are rotated through 135°. [2 marks]

11. After the axes are rotated through 30°, the equation of a curve becomes $X^2 - Y^2 = 6$. Find the original equation. [3 marks]

12. Find the angle through which the axes must be rotated to remove the xy term from $-2x^2 - 6\sqrt{3}xy + 4y^2 = 6$. [1 mark]

13. Transform $4x^2 - 4xy + 7y^2 = -16$ by rotating the axes so that the xy term disappears. [5 marks]

14. Find the angle through which the axes must be rotated so that the line $x + y + 8 = 0$ takes the form $X = \text{constant}$. [1 mark]

15. Find the angle of rotation that reduces the line $\sqrt{3}x + y + 10 = 0$ to the form $X = \text{constant}$, and find the new equation. [3 marks]

16. The axes are rotated through $60°$ and $-2x^2 - 4xy + 3y^2 + 6x + y - 2 = 0$ becomes $AX^2 + 2HXY + BY^2 + 2GX + 2FY + C = 0$. Find C. [1 mark]

17. A rotation of the axes removes the xy term from $3x^2 + 6xy - 5y^2 = -1$ and gives $AX^2 + BY^2 = -1$. Use the invariants to find A and B. [3 marks]

18. The origin is shifted to $(1, 4)$ and the axes are then rotated through $180°$. Find the new coordinates of $(7, 6)$. [1 mark]

19. The origin is shifted to $(2, 1)$ and the axes are then rotated through $45°$. A point has new coordinates $(7, -5)$. Find its original coordinates. [2 marks]

20. Shift the origin to $(1, 4)$ and then rotate the axes through $\tan^{-1}\frac{1}{2}$. Find the transformed equation of $9x^2 + 4xy + 6y^2 - 34x - 52y + 109 = 0$. [5 marks]

Key sheet

Q	Answer
1	$(-2, 12)$
2	$3X^2 + XY + 2Y^2 - 23X - 12Y + 40 = 0$
3	$4x^2 - 5y^2 - 32x - 10y + 39 = 0$
4	$(-3, 5)$
5	$(1, -1)$; $-X^2 + 6XY - Y^2 + 13 = 0$
6	$2G = -13,\ 2F = -20,\ C = 51$
7	New origin $(4, -3)$; $X^2 - Y^2 = 0$
8	$(-1, -3)$
9	$(-\frac{\sqrt{2}}{2}, -\frac{7\sqrt{2}}{2})$
10	$X^2 - Y^2 = 10$
11	$x^2 + 2\sqrt{3}xy - y^2 = 12$
12	$\theta = \frac{\pi}{6}$
13	$\theta = \tan^{-1}\frac{1}{2}$; $3X^2 + 8Y^2 = -16$
14	45°
15	30°; $X = -5$
16	$C = -2$
17	4 and -6 (in some order)
18	$(-6, -2)$
19	$(2 + 6\sqrt{2}, 1 + \sqrt{2})$
20	$10X^2 + 5Y^2 = 12$

Solutions

1. The origin is shifted to $(6, 6)$. A point has new coordinates $(-8, 6)$. Find its original coordinates. [1]

$x = X + h,\ y = Y + k$

$x = -8 + (6) = -2,\ y = 6 + (6) = 12$

Ans: $(-2, 12)$

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2. Find the transformed equation of $3x^2 + xy + 2y^2 + 5x + 8y - 4 = 0$ when the origin is shifted to $(-4, -4)$. [2]

Put $x = X - 4,\ y = Y - 4$ in the equation.

Expanding and collecting terms: $3X^2 + XY + 2Y^2 - 23X - 12Y + 40 = 0$

Ans: $3X^2 + XY + 2Y^2 - 23X - 12Y + 40 = 0$

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3. When the origin is shifted to $(4, -1)$, the equation of a curve becomes $4X^2 - 5Y^2 = 20$. Find its original equation. [2]

Replace X by $x - 4$ and Y by $y + 1$.

$4x^2 - 5y^2 - 32x - 10y + 39 = 0$

Ans: $4x^2 - 5y^2 - 32x - 10y + 39 = 0$

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4. Find the point to which the origin must be shifted to remove the first degree terms from $x^2 + 4xy + 2y^2 - 14x - 8y - 2 = 0$. [3]

$a = 1,\ h = 2,\ b = 2,\ g = -7,\ f = -4,\ c = -2$

Solve $ax + hy + g = 0$ and $hx + by + f = 0$: $x + 2y - 7 = 0$, $2x + 2y - 4 = 0$

So $(x_1, y_1) = (-3, 5)$.

Ans: $(-3, 5)$

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5. Shift the origin to a suitable point so that the equation $-x^2 + 6xy - y^2 + 8x - 8y + 5 = 0$ has no first degree terms. Find the point and the transformed equation. [5]

$a = -1,\ h = 3,\ b = -1,\ g = 4,\ f = -4,\ c = 5$

Solve $ax + hy + g = 0$ and $hx + by + f = 0$: $-x + 3y + 4 = 0$, $3x - y - 4 = 0$

So $(x_1, y_1) = (1, -1)$.

Constant $= gx_1 + fy_1 + c = (4)(1) + (-4)(-1) + (5) = 13$

Transformed equation: $-X^2 + 6XY - Y^2 + 13 = 0$

Ans: $(1, -1)$; $-X^2 + 6XY - Y^2 + 13 = 0$

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6. When the origin is shifted to $(-2, -3)$, the equation $2x^2 + 3y^2 - 5x - 2y = 0$ becomes $AX^2 + 2HXY + BY^2 + 2GX + 2FY + C = 0$. Find $2G$, $2F$ and C. [2]

$2G = \left(\frac{\partial S}{\partial x}\right)_{(-2,-3)} = 2(2)(-2) + (0)(-3) + (-5) = -13$

$2F = \left(\frac{\partial S}{\partial y}\right)_{(-2,-3)} = (0)(-2) + 2(3)(-3) + (-2) = -20$

$C = S(-2, -3) = 51$

Ans: $2G = -13,\ 2F = -20,\ C = 51$

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7. Show that $x^2 - y^2 - 8x - 6y + 7 = 0$ can be reduced to the form $aX^2 + 2hXY + bY^2 = 0$ by a translation of axes. Find the new origin and the reduced equation. [5]

$a = 1,\ h = 0,\ b = -1,\ g = -4,\ f = -3,\ c = 7$

$abc + 2fgh - af^2 - bg^2 - ch^2 = -7 + 0 - 9 + 16 - 0 = 0$

So the reduction is possible.

Solve $x - 4 = 0$ and $-y - 3 = 0$: new origin $(4, -3)$.

Constant $gx_1 + fy_1 + c = 0$, so the reduced equation is $X^2 - Y^2 = 0$

Ans: New origin $(4, -3)$; $X^2 - Y^2 = 0$

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8. The axes are rotated through $180°$. Find the new coordinates of the point $(1, 3)$. [1]

$X = x\cos\theta + y\sin\theta = -1,\ Y = -x\sin\theta + y\cos\theta = -3$

Ans: $(-1, -3)$

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9. The axes are rotated through $45°$ in the clockwise direction. Find the new coordinates of the point $(-4, -3)$. [2]

Clockwise: use $\theta = -45°$.

$X = x\cos\theta + y\sin\theta = -\frac{\sqrt{2}}{2},\ Y = -x\sin\theta + y\cos\theta = -\frac{7\sqrt{2}}{2}$

Ans: $(-\frac{\sqrt{2}}{2}, -\frac{7\sqrt{2}}{2})$

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10. Find the transformed equation of $xy = -5$ when the axes are rotated through $135°$. [2]

Put $x = X\cos\theta - Y\sin\theta = -\frac{\sqrt{2}X}{2} - \frac{\sqrt{2}Y}{2},\ y = X\sin\theta + Y\cos\theta = \frac{\sqrt{2}X}{2} - \frac{\sqrt{2}Y}{2}$.

Simplifying: $X^2 - Y^2 = 10$

Ans: $X^2 - Y^2 = 10$

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11. After the axes are rotated through $30°$, the equation of a curve becomes $X^2 - Y^2 = 6$. Find the original equation. [3]

Replace X by $x\cos\theta + y\sin\theta = \frac{\sqrt{3}x}{2} + \frac{y}{2}$ and Y by $-x\sin\theta + y\cos\theta = -\frac{x}{2} + \frac{\sqrt{3}y}{2}$.

Simplifying: $x^2 + 2\sqrt{3}xy - y^2 = 12$

Ans: $x^2 + 2\sqrt{3}xy - y^2 = 12$

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12. Find the angle through which the axes must be rotated to remove the xy term from $-2x^2 - 6\sqrt{3}xy + 4y^2 = 6$. [1]

$\tan 2\theta = \frac{2h}{a-b} = \frac{-6\sqrt{3}}{-2-(4)} = \sqrt{3}$

$\theta = \frac{\pi}{6} = \frac{\pi}{6}$

Ans: $\theta = \frac{\pi}{6}$

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13. Transform $4x^2 - 4xy + 7y^2 = -16$ by rotating the axes so that the xy term disappears. [5]

$\tan 2\theta = \frac{2h}{a-b} = \frac{4}{3}$, so $\cos\theta = \frac{2\sqrt{5}}{5},\ \sin\theta = \frac{\sqrt{5}}{5}$.

Put $x = \frac{2\sqrt{5}X}{5} - \frac{\sqrt{5}Y}{5},\ y = \frac{\sqrt{5}X}{5} + \frac{2\sqrt{5}Y}{5}$.

The XY terms cancel and the equation becomes $3X^2 + 8Y^2 = -16$

Check: $A + B = 11 = a + b$.

Ans: $\theta = \tan^{-1}\frac{1}{2}$; $3X^2 + 8Y^2 = -16$

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14. Find the angle through which the axes must be rotated so that the line $x + y + 8 = 0$ takes the form $X = \text{constant}$. [1]

Angle $= \tan^{-1}\left(\frac{b}{a}\right) = 45°$.

Ans: $45°$

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15. Find the angle of rotation that reduces the line $\sqrt{3}x + y + 10 = 0$ to the form $X = \text{constant}$, and find the new equation. [3]

Angle $= \tan^{-1}\left(\frac{b}{a}\right) = 30°$.

Here $\cos\theta = \frac{\sqrt{3}}{2}$, $\sin\theta = \frac{1}{2}$; put $x = \frac{\sqrt{3}X}{2} - \frac{Y}{2}$, $y = \frac{X}{2} + \frac{\sqrt{3}Y}{2}$.

The line becomes $2X + 10 = 0$, that is $X = -5$

Ans: $30°$; $X = -5$

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16. The axes are rotated through $60°$ and $-2x^2 - 4xy + 3y^2 + 6x + y - 2 = 0$ becomes $AX^2 + 2HXY + BY^2 + 2GX + 2FY + C = 0$. Find C. [1]

$C = c$ (an invariant under rotation).

$= -2$

Ans: $C = -2$

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17. A rotation of the axes removes the xy term from $3x^2 + 6xy - 5y^2 = -1$ and gives $AX^2 + BY^2 = -1$. Use the invariants to find A and B. [3]

$A + B = a + b = -2$

$H^2 - AB = h^2 - ab$ with $H = 0$: $AB = ab - h^2 = -24$

So A and B are the roots of $t^2 - (-2)t + (-24) = 0$: 4 and -6 (in some order).

Ans: 4 **and** -6 **(in some order)**

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18. The origin is shifted to $(1, 4)$ and the axes are then rotated through $180°$. Find the new coordinates of $(7, 6)$. [1]

$x - h = 6,\ y - k = 2$

$X = (x - h)\cos\theta + (y - k)\sin\theta = -6$

$Y = -(x - h)\sin\theta + (y - k)\cos\theta = -2$

Ans: $(-6, -2)$

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19. The origin is shifted to $(2, 1)$ and the axes are then rotated through $45°$. A point has new coordinates $(7, -5)$. Find its original coordinates. [2]

$x - h = X\cos\theta - Y\sin\theta = 6\sqrt{2}$

$y - k = X\sin\theta + Y\cos\theta = \sqrt{2}$

So $(x, y) = (2 + 6\sqrt{2}, 1 + \sqrt{2})$.

Ans: $(2 + 6\sqrt{2}, 1 + \sqrt{2})$

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20. Shift the origin to $(1, 4)$ and then rotate the axes through $\tan^{-1}\frac{1}{2}$. Find the transformed equation of $9x^2 + 4xy + 6y^2 - 34x - 52y + 109 = 0$. [5]

Shift the origin: $x - h = x - 1,\ y - k = y - 4$. The equation is $9(x-1)^2 + 4(x-1)(y-4) + 6(y-4)^2 = 12$.

Rotate: $\cos\theta = \frac{2\sqrt{5}}{5},\ \sin\theta = \frac{\sqrt{5}}{5}$; put $x - h = \frac{2\sqrt{5}X}{5} - \frac{\sqrt{5}Y}{5},\ y - k = \frac{\sqrt{5}X}{5} + \frac{2\sqrt{5}Y}{5}$.

The XY terms cancel: $10X^2 + 5Y^2 = 12$

Ans: $10X^2 + 5Y^2 = 12$