EXPRESSWALA ACADEMY

Maths 1A: Algebra of Matrices

Question Paper, Set 09

Time: 1 hour 30 minutes | Maximum marks: 50

Name: _____ Class: _____ Date: _____

1. Answer all 20 questions. The marks for each question are shown in brackets.
2. Show your working for the 2, 3 and 5 mark questions.
3. The key sheet and worked solutions are at the end of this paper.

1. Write the order of $A = \begin{bmatrix} 2 & 6 & 7 \end{bmatrix}$ and the element a_{13}. [1 mark]

2. Construct the 3×3 matrix $A = [a_{ij}]$ with $a_{ij} = i^2 + j$. [2 marks]

3. Name the type of the square matrix $\begin{bmatrix} -2 & 2 & 7 \\ 0 & -1 & -5 \\ 0 & 0 & 2 \end{bmatrix}$ (the most special type that fits). [1 mark]

4. If $A = \begin{bmatrix} 4 & 1 \\ 6 & 3 \end{bmatrix}$, find $\operatorname{tr}(-3I_3) + \operatorname{tr}(A^T)$. [1 mark]

5. Find x and y if $\begin{bmatrix} 2x + y & 5 \\ 10 & x - y \end{bmatrix} = \begin{bmatrix} 11 & 5 \\ 10 & 1 \end{bmatrix}$. [2 marks]

6. If $A = \begin{bmatrix} -3 & 7 & -5 \\ -6 & -1 & -4 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & -1 & 0 \\ 0 & 3 & -5 \end{bmatrix}$, find $3A - 4B$. [2 marks]

7. Find the matrices X and Y if $X + Y = \begin{bmatrix} 6 & -7 \\ 2 & 4 \end{bmatrix}$ and $X - Y = \begin{bmatrix} 2 & 1 \\ 2 & -6 \end{bmatrix}$. [3 marks]

8. If $A = \begin{bmatrix} -4 & -1 \\ -1 & 0 \\ 3 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} -1 \\ 4 \end{bmatrix}$, find AB. [2 marks]

9. If $A = \begin{bmatrix} -3 & 1 & 2 \\ 2 & -1 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 4 \\ 0 & 2 \\ -1 & -3 \end{bmatrix}$, find AB and BA. [3 marks]

10. If $A = \begin{bmatrix} 1 & 2 & -2 \\ 2 & 3 & 1 \\ -1 & 2 & 2 \end{bmatrix}$, find $A^2 - 5A - 5I$. [5 marks]

11. Write the transpose of $A = \begin{bmatrix} 7 \\ 7 \\ 5 \\ 8 \end{bmatrix}$ and its order. [1 mark]

12. If $A = \begin{bmatrix} -2 & -2 & 1 \\ 2 & -1 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} -3 & -1 \\ 3 & 2 \\ 0 & -1 \end{bmatrix}$, verify that $(AB)^T = B^T A^T$. [3 marks]

13. Find x if $\begin{bmatrix} 2 & x+10 & -7 \\ -8 & -8 & 3 \\ -7 & 3 & -4 \end{bmatrix}$ is symmetric. [1 mark]

14. Find a, b, c if $\begin{bmatrix} 0 & 1 & b \\ a & 0 & 5 \\ -9 & c & 0 \end{bmatrix}$ is skew-symmetric. [2 marks]

15. Express $A = \begin{bmatrix} 3 & -4 & 1 \\ 5 & 7 & 6 \\ -6 & -5 & 0 \end{bmatrix}$ as the sum of a symmetric and a skew-symmetric matrix. [5 marks]

16. Show that $A = \frac{1}{3}\begin{bmatrix} -2 & -2 & 1 \\ -1 & 2 & 2 \\ 2 & -1 & 2 \end{bmatrix}$ is an orthogonal matrix. [5 marks]

17. Show that $A = \begin{bmatrix} -2 & 1 \\ -4 & 2 \end{bmatrix}$ is nilpotent and find its index. [2 marks]

18. Find the conjugate $\bar{A}$ of $A = \begin{bmatrix} 1+i & 5-2i \\ -6-3i & 7i-1 \end{bmatrix}$. [1 mark]

19. For $A = \begin{bmatrix} 1 & -5 \\ 2 & -4 \end{bmatrix}$, find numbers p and q with $A^3 = pA + qI$, using the equation that A satisfies. [3 marks]

20. Find the trace, the sum of the 2×2 principal minors and $|A|$ for $A = \begin{bmatrix} 3 & 0 & 4 \\ -2 & 1 & -3 \\ -3 & -1 & 0 \end{bmatrix}$, and write the cubic equation that A satisfies. [5 marks]

Key sheet

Q	Answer
1	Order 1×3; $a_{13} = 7$
2	$A = \begin{bmatrix} 2 & 3 & 4 \\ 5 & 6 & 7 \\ 10 & 11 & 12 \end{bmatrix}$
3	An upper triangular matrix
4	-2
5	$x = 4$, $y = 3$
6	$\begin{bmatrix} -5 & 25 & -15 \\ -18 & -15 & 8 \end{bmatrix}$
7	$X = \begin{bmatrix} 4 & -3 \\ 2 & -1 \end{bmatrix}$, $Y = \begin{bmatrix} 2 & -4 \\ 0 & 5 \end{bmatrix}$
8	$AB = \begin{bmatrix} 0 \\ 1 \\ -3 \end{bmatrix}$
9	$AB = \begin{bmatrix} -8 & -16 \\ 2 & 0 \end{bmatrix}$, $BA = \begin{bmatrix} 2 & -2 & 12 \\ 4 & -2 & 4 \\ -3 & 2 & -8 \end{bmatrix}$
10	$\begin{bmatrix} -3 & -6 & 6 \\ -3 & -5 & -4 \\ 6 & -2 & -7 \end{bmatrix}$
11	$A^T = \begin{bmatrix} 7 & 7 & 5 & 8 \end{bmatrix}$, order 1×4
12	Both equal $\begin{bmatrix} 0 & -9 \\ -3 & -5 \end{bmatrix}$
13	$x = -18$
14	$a = -1$, $b = 9$, $c = -5$
15	$\begin{bmatrix} 3 & \frac{1}{2} & -\frac{5}{2} \\ \frac{1}{2} & 7 & \frac{1}{2} \\ -\frac{5}{2} & \frac{1}{2} & 0 \end{bmatrix} + \begin{bmatrix} 0 & -\frac{9}{2} & \frac{7}{2} \\ \frac{9}{2} & 0 & \frac{11}{2} \\ -\frac{7}{2} & -\frac{11}{2} & 0 \end{bmatrix}$
16	$AA^T = A^TA = I$, so A is orthogonal
17	Nilpotent of index 2
18	$\bar{A} = \begin{bmatrix} 1 - i & 5 + 2i \\ 3i - 6 & -1 - 7i \end{bmatrix}$
19	$p = 3$, $q = 18$
20	$A^3 - 4A^2 + 12A - 11I = O$

Solutions

1. Write the order of $A = \begin{bmatrix} 2 & 6 & 7 \end{bmatrix}$ and the element a_{13}. [1]

A has 1 rows and 3 columns, so its order is 1×3.

a_{13} is in row 1, column 3: $a_{13} = 7$.

Ans: Order 1×3; $a_{13} = 7$

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2. Construct the 3×3 matrix $A = [a_{ij}]$ with $a_{ij} = i^2 + j$. [2]

$a_{ij} = i^2 + j$

$a_{11} = 2$, $a_{12} = 3$, $a_{13} = 4$

$a_{21} = 5$, $a_{22} = 6$, $a_{23} = 7$

$a_{31} = 10$, $a_{32} = 11$, $a_{33} = 12$

$$A = \begin{bmatrix} 2 & 3 & 4 \\ 5 & 6 & 7 \\ 10 & 11 & 12 \end{bmatrix}$$

Ans: $A = \begin{bmatrix} 2 & 3 & 4 \\ 5 & 6 & 7 \\ 10 & 11 & 12 \end{bmatrix}$

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3. Name the type of the square matrix $\begin{bmatrix} -2 & 2 & 7 \\ 0 & -1 & -5 \\ 0 & 0 & 2 \end{bmatrix}$ (the most special type that fits). [1]

Here every element below the principal diagonal is 0 ($a_{ij} = 0$ for $i > j$).

So it is an upper triangular matrix.

Ans: An upper triangular matrix

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4. If $A = \begin{bmatrix} 4 & 1 \\ 6 & 3 \end{bmatrix}$, find $\operatorname{tr}(-3I_3) + \operatorname{tr}(A^T)$. [1]

$\operatorname{tr}(-3I_3) = -3 \times 3 = -9$

$\operatorname{tr}(A^T) = \operatorname{tr}(A) = 7$

Sum $= -2$

Ans: -2

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5. Find x and y if $\begin{bmatrix} 2x + y & 5 \\ 10 & x - y \end{bmatrix} = \begin{bmatrix} 11 & 5 \\ 10 & 1 \end{bmatrix}$. [2]

Equate corresponding elements:

$2x + y = 11, \quad x - y = 1$

Solving, $x = 4$, $y = 3$.

Ans: $x = 4$, $y = 3$

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6. If $A = \begin{bmatrix} -3 & 7 & -5 \\ -6 & -1 & -4 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & -1 & 0 \\ 0 & 3 & -5 \end{bmatrix}$, find $3A - 4B$. [2]

$3A = \begin{bmatrix} -9 & 21 & -15 \\ -18 & -3 & -12 \end{bmatrix}$, $4B = \begin{bmatrix} -4 & -4 & 0 \\ 0 & 12 & -20 \end{bmatrix}$

$3A - 4B = \begin{bmatrix} -5 & 25 & -15 \\ -18 & -15 & 8 \end{bmatrix}$

Ans: $\begin{bmatrix} -5 & 25 & -15 \\ -18 & -15 & 8 \end{bmatrix}$

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7. Find the matrices X and Y if $X + Y = \begin{bmatrix} 6 & -7 \\ 2 & 4 \end{bmatrix}$ and $X - Y = \begin{bmatrix} 2 & 1 \\ 2 & -6 \end{bmatrix}$. [3]

Adding, $2X = \begin{bmatrix} 8 & -6 \\ 4 & -2 \end{bmatrix}$, so $X = \begin{bmatrix} 4 & -3 \\ 2 & -1 \end{bmatrix}$.

Subtracting, $2Y = \begin{bmatrix} 4 & -8 \\ 0 & 10 \end{bmatrix}$, so $Y = \begin{bmatrix} 2 & -4 \\ 0 & 5 \end{bmatrix}$.

Ans: $X = \begin{bmatrix} 4 & -3 \\ 2 & -1 \end{bmatrix}$, $Y = \begin{bmatrix} 2 & -4 \\ 0 & 5 \end{bmatrix}$

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8. If $A = \begin{bmatrix} -4 & -1 \\ -1 & 0 \\ 3 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} -1 \\ 4 \end{bmatrix}$, find AB. [2]

A is 3×2 and B is 2×1, so AB is 3×1.

$c_{11} = (-4)(-1) + (-1)(4) = 0$

$c_{21} = (-1)(-1) + (0)(4) = 1$

$c_{31} = (3)(-1) + (0)(4) = -3$

$AB = \begin{bmatrix} 0 \\ 1 \\ -3 \end{bmatrix}$

Ans: $AB = \begin{bmatrix} 0 \\ 1 \\ -3 \end{bmatrix}$

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9. If $A = \begin{bmatrix} -3 & 1 & 2 \\ 2 & -1 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 4 \\ 0 & 2 \\ -1 & -3 \end{bmatrix}$, find AB and BA. [3]

AB is 2×2 and BA is 3×3.

$AB = \begin{bmatrix} -8 & -16 \\ 2 & 0 \end{bmatrix}$

$BA = \begin{bmatrix} 2 & -2 & 12 \\ 4 & -2 & 4 \\ -3 & 2 & -8 \end{bmatrix}$

The two products are not even of the same order.

Ans: $AB = \begin{bmatrix} -8 & -16 \\ 2 & 0 \end{bmatrix}$, $BA = \begin{bmatrix} 2 & -2 & 12 \\ 4 & -2 & 4 \\ -3 & 2 & -8 \end{bmatrix}$

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10. If $A = \begin{bmatrix} 1 & 2 & -2 \\ 2 & 3 & 1 \\ -1 & 2 & 2 \end{bmatrix}$, find $A^2 - 5A - 5I$. [5]

$A^2 = A \cdot A = \begin{bmatrix} 7 & 4 & -4 \\ 7 & 15 & 1 \\ 1 & 8 & 8 \end{bmatrix}$

$A^2 - 5A - 5I = \begin{bmatrix} 7 & 4 & -4 \\ 7 & 15 & 1 \\ 1 & 8 & 8 \end{bmatrix} - \begin{bmatrix} 5 & 10 & -10 \\ 10 & 15 & 5 \\ -5 & 10 & 10 \end{bmatrix} - \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix}$

$= \begin{bmatrix} -3 & -6 & 6 \\ -3 & -5 & -4 \\ 6 & -2 & -7 \end{bmatrix}$

Ans: $\begin{bmatrix} -3 & -6 & 6 \\ -3 & -5 & -4 \\ 6 & -2 & -7 \end{bmatrix}$

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11. Write the transpose of $A = \begin{bmatrix} 7 \\ 7 \\ 5 \\ 8 \end{bmatrix}$ and its order. [1]

Rows of A become columns of A^T.

$A^T = \begin{bmatrix} 7 & 7 & 5 & 8 \end{bmatrix}$

Order 1×4.

Ans: $A^T = \begin{bmatrix} 7 & 7 & 5 & 8 \end{bmatrix}$, **order** 1×4

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12. If $A = \begin{bmatrix} -2 & -2 & 1 \\ 2 & -1 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} -3 & -1 \\ 3 & 2 \\ 0 & -1 \end{bmatrix}$, verify that $(AB)^T = B^T A^T$. [3]

$AB = \begin{bmatrix} 0 & -3 \\ -9 & -5 \end{bmatrix}$

$(AB)^T = \begin{bmatrix} 0 & -9 \\ -3 & -5 \end{bmatrix}$

$B^T = \begin{bmatrix} -3 & 3 & 0 \\ -1 & 2 & -1 \end{bmatrix}$, $A^T = \begin{bmatrix} -2 & 2 \\ -2 & -1 \\ 1 & 1 \end{bmatrix}$

$B^T A^T = \begin{bmatrix} 0 & -9 \\ -3 & -5 \end{bmatrix}$

So $(AB)^T = B^T A^T$.

Ans: Both equal $\begin{bmatrix} 0 & -9 \\ -3 & -5 \end{bmatrix}$

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13. Find x if $\begin{bmatrix} 2 & x+10 & -7 \\ -8 & -8 & 3 \\ -7 & 3 & -4 \end{bmatrix}$ is symmetric. [1]

Symmetric means $a_{ij} = a_{ji}$.

$a_{12} = a_{21} \Rightarrow x + 10 = -8$

$x = -18$

Ans: $x = -18$

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14. Find a, b, c if $\begin{bmatrix} 0 & 1 & b \\ a & 0 & 5 \\ -9 & c & 0 \end{bmatrix}$ is skew-symmetric. [2]

Skew-symmetric means $a_{ij} = -a_{ji}$.

$a = -(1) = -1, \quad b = -(-9) = 9, \quad c = -(5) = -5$

Ans: $a = -1$, $b = 9$, $c = -5$

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15. Express $A = \begin{bmatrix} 3 & -4 & 1 \\ 5 & 7 & 6 \\ -6 & -5 & 0 \end{bmatrix}$ as the sum of a symmetric and a skew-symmetric matrix. [5]

$A^T = \begin{bmatrix} 3 & 5 & -6 \\ -4 & 7 & -5 \\ 1 & 6 & 0 \end{bmatrix}$

$A + A^T = \begin{bmatrix} 6 & 1 & -5 \\ 1 & 14 & 1 \\ -5 & 1 & 0 \end{bmatrix}, \quad P = \frac{1}{2}(A + A^T) = \begin{bmatrix} 3 & \frac{1}{2} & -\frac{5}{2} \\ \frac{1}{2} & 7 & \frac{1}{2} \\ -\frac{5}{2} & \frac{1}{2} & 0 \end{bmatrix}$

$A - A^T = \begin{bmatrix} 0 & -9 & 7 \\ 9 & 0 & 11 \\ -7 & -11 & 0 \end{bmatrix}, \quad Q = \frac{1}{2}(A - A^T) = \begin{bmatrix} 0 & -\frac{9}{2} & \frac{7}{2} \\ \frac{9}{2} & 0 & \frac{11}{2} \\ -\frac{7}{2} & -\frac{11}{2} & 0 \end{bmatrix}$

$P^T = P$ and $Q^T = -Q$, and $P + Q = A$.

$A = \begin{bmatrix} 3 & \frac{1}{2} & -\frac{5}{2} \\ \frac{1}{2} & 7 & \frac{1}{2} \\ -\frac{5}{2} & \frac{1}{2} & 0 \end{bmatrix} + \begin{bmatrix} 0 & -\frac{9}{2} & \frac{7}{2} \\ \frac{9}{2} & 0 & \frac{11}{2} \\ -\frac{7}{2} & -\frac{11}{2} & 0 \end{bmatrix}$

Ans: $\begin{bmatrix} 3 & \frac{1}{2} & -\frac{5}{2} \\ \frac{1}{2} & 7 & \frac{1}{2} \\ -\frac{5}{2} & \frac{1}{2} & 0 \end{bmatrix} + \begin{bmatrix} 0 & -\frac{9}{2} & \frac{7}{2} \\ \frac{9}{2} & 0 & \frac{11}{2} \\ -\frac{7}{2} & -\frac{11}{2} & 0 \end{bmatrix}$

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16. Show that $A = \frac{1}{3}\begin{bmatrix} -2 & -2 & 1 \\ -1 & 2 & 2 \\ 2 & -1 & 2 \end{bmatrix}$ is an orthogonal matrix. [5]

$A^T = \frac{1}{3}\begin{bmatrix} -2 & -1 & 2 \\ -2 & 2 & -1 \\ 1 & 2 & 2 \end{bmatrix}$

$AA^T = \frac{1}{9}\begin{bmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{bmatrix}$

$= \frac{1}{9} \cdot 9I = I$

Similarly $A^T A = I$, so A is orthogonal.

Ans: $AA^T = A^T A = I$, **so A is orthogonal**

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17. Show that $A = \begin{bmatrix} -2 & 1 \\ -4 & 2 \end{bmatrix}$ is nilpotent and find its index. [2]

$A \neq O, \quad A^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = O$

So A is nilpotent of index 2.

Ans: Nilpotent of index 2

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18. Find the conjugate $\bar{A}$ of $A = \begin{bmatrix} 1+i & 5-2i \\ -6-3i & 7i-1 \end{bmatrix}$. [1]

Replace each element by its complex conjugate.

$\bar{A} = \begin{bmatrix} 1-i & 5+2i \\ 3i-6 & -1-7i \end{bmatrix}$

Ans: $\bar{A} = \begin{bmatrix} 1-i & 5+2i \\ 3i-6 & -1-7i \end{bmatrix}$

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19. For $A = \begin{bmatrix} 1 & -5 \\ 2 & -4 \end{bmatrix}$, find numbers p and q with $A^3 = pA + qI$, using the equation that A satisfies. [3]

$A^2 - (a+d)A + (ad-bc)I = O \Rightarrow A^2 = -3A - 6I$

$A^3 = A \cdot A^2 = -3A^2 - 6A = -3(-3A - 6I) - 6A$

$= 3A + 18I$

So $p = 3$, $q = 18$.

Ans: $p = 3$, $q = 18$

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20. Find the trace, the sum of the 2×2 principal minors and $|A|$ for $A = \begin{bmatrix} 3 & 0 & 4 \\ -2 & 1 & -3 \\ -3 & -1 & 0 \end{bmatrix}$, and write the cubic equation that A satisfies. [5]

$\operatorname{tr} A = 3 + 1 + 0 = 4$

$\begin{vmatrix} 1 & -3 \\ -1 & 0 \end{vmatrix} = -3, \quad \begin{vmatrix} 3 & 4 \\ -3 & 0 \end{vmatrix} = 12, \quad \begin{vmatrix} 3 & 0 \\ -2 & 1 \end{vmatrix} = 3$

Sum of the principal minors $= 12$.

$|A| = 11$

So $A^3 - 4A^2 + 12A - 11I = O$.

Ans: $A^3 - 4A^2 + 12A - 11I = O$
