EXPRESSWALA ACADEMY

Maths 1A: Algebra of Matrices

Question Paper, Set 07

Time: 1 hour 30 minutes | Maximum marks: 50

Name: _____ Class: _____ Date: _____

1. Answer all 20 questions. The marks for each question are shown in brackets.
2. Show your working for the 2, 3 and 5 mark questions.
3. The key sheet and worked solutions are at the end of this paper.

1. Write the order of $A = \begin{bmatrix} 6 & 8 & 10 & -7 \\ -8 & -1 & -5 & 0 \\ -7 & -5 & -8 & -6 \end{bmatrix}$ and the element a_{31}. [1 mark]

2. Construct the 2×2 matrix $A = [a_{ij}]$ with $a_{ij} = ij - 4$. [2 marks]

3. Name the type of the square matrix $\begin{bmatrix} 4 & 0 \\ 0 & 5 \end{bmatrix}$ (the most special type that fits). [1 mark]

4. Find the trace of $\begin{bmatrix} -8 & -1 & 1 \\ 6 & 6 & -3 \\ -2 & 0 & 0 \end{bmatrix}$. [1 mark]

5. Find x and y if $\begin{bmatrix} x + 2y & 5 \\ 8 & 3x + 2y \end{bmatrix} = \begin{bmatrix} 9 & 5 \\ 8 & 3 \end{bmatrix}$. [2 marks]

6. If $A = \begin{bmatrix} 3 & -6 & 0 \\ -2 & 4 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & 5 & -1 \\ 5 & -6 & -3 \end{bmatrix}$, find $2A - 3B$. [2 marks]

7. Find the matrices X and Y if $X + Y = \begin{bmatrix} -4 & 6 \\ 4 & -2 \end{bmatrix}$ and $X - Y = \begin{bmatrix} -2 & 0 \\ 0 & -8 \end{bmatrix}$. [3 marks]

8. If $A = \begin{bmatrix} 0 & 4 & 1 \\ -4 & 2 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}$, find AB. [2 marks]

9. If $A = \begin{bmatrix} -3 & 1 & -3 \\ 3 & 3 & -3 \end{bmatrix}$ and $B = \begin{bmatrix} -3 & 1 \\ -1 & 4 \\ -2 & -1 \end{bmatrix}$, find AB and BA. [3 marks]

10. If $A = \begin{bmatrix} 0 & 3 & 2 \\ 3 & -2 & -2 \\ -1 & 2 & -2 \end{bmatrix}$, find $A^2 - 3A - 5I$. [5 marks]

11. Write the transpose of $A = \begin{bmatrix} 10 & 1 & 3 & -3 \\ 5 & -5 & 5 & -6 \end{bmatrix}$ and its order. [1 mark]

12. If $A = \begin{bmatrix} 3 & -1 & 0 \\ -1 & 4 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & 2 \\ -1 & -3 \\ 3 & 4 \end{bmatrix}$, verify that $(AB)^T = B^T A^T$. [3 marks]

13. Find x if $\begin{bmatrix} 9 & -8 & -3 \\ -8 & -8 & -4x-3 \\ -3 & 1 & -7 \end{bmatrix}$ is symmetric. [1 mark]

14. Find a, b, c if $\begin{bmatrix} 0 & 6 & b \\ a & 0 & -8 \\ 1 & c & 0 \end{bmatrix}$ is skew-symmetric. [2 marks]

15. Express $A = \begin{bmatrix} 6 & -6 & 2 \\ -4 & 4 & -5 \\ 3 & 6 & 1 \end{bmatrix}$ as the sum of a symmetric and a skew-symmetric matrix. [5 marks]

16. Show that $A = \frac{1}{9}\begin{bmatrix} 8 & 1 & 4 \\ 4 & -4 & -7 \\ 1 & 8 & -4 \end{bmatrix}$ is an orthogonal matrix. [5 marks]

17. Show that $A = \begin{bmatrix} 2 & 1 \\ -2 & -1 \end{bmatrix}$ is idempotent. [2 marks]

18. Find the conjugate $\bar{A}$ of $A = \begin{bmatrix} 3-i & 2+5i \\ 3-i & -5-4i \end{bmatrix}$. [1 mark]

19. For $A = \begin{bmatrix} 6 & -1 \\ 6 & -5 \end{bmatrix}$, find numbers p and q with $A^3 = pA + qI$, using the equation that A satisfies. [3 marks]

20. Find the trace, the sum of the 2×2 principal minors and $|A|$ for $A = \begin{bmatrix} 4 & -1 & 3 \\ 3 & -1 & -1 \\ -3 & 3 & -2 \end{bmatrix}$, and write the cubic equation that A satisfies. [5 marks]

Key sheet

Q	Answer
1	Order 3×4; $a_{31} = -7$
2	$A = \begin{bmatrix} -3 & -2 \\ -2 & 0 \end{bmatrix}$
3	A diagonal matrix (but not a scalar matrix)
4	-2
5	$x = -3$, $y = 6$
6	$\begin{bmatrix} 9 & -27 & 3 \\ -19 & 26 & 9 \end{bmatrix}$
7	$X = \begin{bmatrix} -3 & 3 \\ 2 & -5 \end{bmatrix}$, $Y = \begin{bmatrix} -1 & 3 \\ 2 & 3 \end{bmatrix}$
8	$AB = \begin{bmatrix} 10 \\ -4 \end{bmatrix}$
9	$AB = \begin{bmatrix} 14 & 4 \\ -6 & 18 \end{bmatrix}$, $BA = \begin{bmatrix} 12 & 0 & 6 \\ 15 & 11 & -9 \\ 3 & -5 & 9 \end{bmatrix}$
10	$\begin{bmatrix} 2 & -11 & -16 \\ -13 & 10 & 20 \\ 11 & -17 & -1 \end{bmatrix}$
11	$A^T = \begin{bmatrix} 10 & 5 \\ 1 & -5 \\ 3 & 5 \\ -3 & -6 \end{bmatrix}$, order 4×2
12	Both equal $\begin{bmatrix} -2 & -3 \\ 9 & -14 \end{bmatrix}$
13	$x = -1$
14	$a = -6$, $b = -1$, $c = 8$
15	$\begin{bmatrix} 6 & -5 & \frac{5}{2} \\ -5 & 4 & \frac{1}{2} \\ \frac{5}{2} & \frac{1}{2} & 1 \end{bmatrix} + \begin{bmatrix} 0 & -1 & -\frac{1}{2} \\ 1 & 0 & -\frac{11}{2} \\ \frac{1}{2} & \frac{11}{2} & 0 \end{bmatrix}$
16	$AA^T = A^T A = I$, so A is orthogonal
17	$A^2 = A$
18	$\bar{A} = \begin{bmatrix} 3+i & 2-5i \\ 3+i & 4i-5 \end{bmatrix}$
19	$p = 25$, $q = 24$
20	$A^3 - A^2 + 5A - 29I = O$

Solutions

1. Write the order of $A = \begin{bmatrix} 6 & 8 & 10 & -7 \\ -8 & -1 & -5 & 0 \\ -7 & -5 & -8 & -6 \end{bmatrix}$ and the element a_{31}. [1]

A has 3 rows and 4 columns, so its order is 3×4.

a_{31} is in row 3, column 1: $a_{31} = -7$.

Ans: Order 3×4; $a_{31} = -7$

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2. Construct the 2×2 matrix $A = [a_{ij}]$ with $a_{ij} = ij - 4$. [2]

$a_{ij} = ij - 4$

$a_{11} = -3$, $a_{12} = -2$

$a_{21} = -2$, $a_{22} = 0$

$A = \begin{bmatrix} -3 & -2 \\ -2 & 0 \end{bmatrix}$

Ans: $A = \begin{bmatrix} -3 & -2 \\ -2 & 0 \end{bmatrix}$

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3. Name the type of the square matrix $\begin{bmatrix} 4 & 0 \\ 0 & 5 \end{bmatrix}$ (the most special type that fits). [1]

Here every element outside the principal diagonal is 0 and the diagonal elements are not all equal. So it is a diagonal matrix (but not a scalar matrix).

Ans: A diagonal matrix (but not a scalar matrix)

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4. Find the trace of $\begin{bmatrix} -8 & -1 & 1 \\ 6 & 6 & -3 \\ -2 & 0 & 0 \end{bmatrix}$. [1]

$\operatorname{tr}(A) = a_{11} + a_{22} + a_{33}$

$= -8 + 6 + 0 = -2$

Ans: -2

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5. Find x and y if $\begin{bmatrix} x + 2y & 5 \\ 8 & 3x + 2y \end{bmatrix} = \begin{bmatrix} 9 & 5 \\ 8 & 3 \end{bmatrix}$. [2]

Equate corresponding elements:

$x + 2y = 9, \quad 3x + 2y = 3$

Solving, $x = -3$, $y = 6$.

Ans: $x = -3$, $y = 6$

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6. If $A = \begin{bmatrix} 3 & -6 & 0 \\ -2 & 4 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & 5 & -1 \\ 5 & -6 & -3 \end{bmatrix}$, find $2A - 3B$. [2]

$2A = \begin{bmatrix} 6 & -12 & 0 \\ -4 & 8 & 0 \end{bmatrix}, \quad 3B = \begin{bmatrix} -3 & 15 & -3 \\ 15 & -18 & -9 \end{bmatrix}$

$2A - 3B = \begin{bmatrix} 9 & -27 & 3 \\ -19 & 26 & 9 \end{bmatrix}$

Ans: $\begin{bmatrix} 9 & -27 & 3 \\ -19 & 26 & 9 \end{bmatrix}$

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7. Find the matrices X and Y if $X+Y=\begin{bmatrix}-4 & 6\\ 4 & -2\end{bmatrix}$ and $X-Y=\begin{bmatrix}-2 & 0\\ 0 & -8\end{bmatrix}$. [3]

Adding, $2X=\begin{bmatrix}-6 & 6\\ 4 & -10\end{bmatrix}$, so $X=\begin{bmatrix}-3 & 3\\ 2 & -5\end{bmatrix}$.

Subtracting, $2Y=\begin{bmatrix}-2 & 6\\ 4 & 6\end{bmatrix}$, so $Y=\begin{bmatrix}-1 & 3\\ 2 & 3\end{bmatrix}$.

Ans: $X=\begin{bmatrix}-3 & 3\\ 2 & -5\end{bmatrix}$, $Y=\begin{bmatrix}-1 & 3\\ 2 & 3\end{bmatrix}$

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8. If $A=\begin{bmatrix}0 & 4 & 1\\ -4 & 2 & 0\end{bmatrix}$ and $B=\begin{bmatrix}2\\ 2\\ 2\end{bmatrix}$, find AB. [2]

A is 2×3 and B is 3×1, so AB is 2×1.

$c_{11}=(0)(2)+(4)(2)+(1)(2)=10$

$c_{21}=(-4)(2)+(2)(2)+(0)(2)=-4$

$AB=\begin{bmatrix}10\\ -4\end{bmatrix}$

Ans: $AB=\begin{bmatrix}10\\ -4\end{bmatrix}$

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9. If $A=\begin{bmatrix}-3 & 1 & -3\\ 3 & 3 & -3\end{bmatrix}$ and $B=\begin{bmatrix}-3 & 1\\ -1 & 4\\ -2 & -1\end{bmatrix}$, find AB and BA. [3]

AB is 2×2 and BA is 3×3.

$AB=\begin{bmatrix}14 & 4\\ -6 & 18\end{bmatrix}$

$BA=\begin{bmatrix}12 & 0 & 6\\ 15 & 11 & -9\\ 3 & -5 & 9\end{bmatrix}$

The two products are not even of the same order.

Ans: $AB=\begin{bmatrix}14 & 4\\ -6 & 18\end{bmatrix}$, $BA=\begin{bmatrix}12 & 0 & 6\\ 15 & 11 & -9\\ 3 & -5 & 9\end{bmatrix}$

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10. If $A=\begin{bmatrix}0 & 3 & 2\\ 3 & -2 & -2\\ -1 & 2 & -2\end{bmatrix}$, find $A^2-3A-5I$. [5]

$A^2=A\cdot A=\begin{bmatrix}7 & -2 & -10\\ -4 & 9 & 14\\ 8 & -11 & -2\end{bmatrix}$

$A^2-3A-5I=\begin{bmatrix}7 & -2 & -10\\ -4 & 9 & 14\\ 8 & -11 & -2\end{bmatrix}-\begin{bmatrix}0 & 9 & 6\\ 9 & -6 & -6\\ -3 & 6 & -6\end{bmatrix}-\begin{bmatrix}5 & 0 & 0\\ 0 & 5 & 0\\ 0 & 0 & 5\end{bmatrix}$

$=\begin{bmatrix}2 & -11 & -16\\ -13 & 10 & 20\\ 11 & -17 & -1\end{bmatrix}$

Ans: $\begin{bmatrix}2 & -11 & -16\\ -13 & 10 & 20\\ 11 & -17 & -1\end{bmatrix}$

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11. Write the transpose of $A = \begin{bmatrix} 10 & 1 & 3 & -3 \\ 5 & -5 & 5 & -6 \end{bmatrix}$ and its order. [1]

Rows of A become columns of A^T.

$$A^T = \begin{bmatrix} 10 & 5 \\ 1 & -5 \\ 3 & 5 \\ -3 & -6 \end{bmatrix}$$

Order 4×2.

Ans: $A^T = \begin{bmatrix} 10 & 5 \\ 1 & -5 \\ 3 & 5 \\ -3 & -6 \end{bmatrix}$, **order** 4×2

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12. If $A = \begin{bmatrix} 3 & -1 & 0 \\ -1 & 4 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & 2 \\ -1 & -3 \\ 3 & 4 \end{bmatrix}$, verify that $(AB)^T = B^T A^T$. [3]

$$AB = \begin{bmatrix} -2 & 9 \\ -3 & -14 \end{bmatrix}$$

$$(AB)^T = \begin{bmatrix} -2 & -3 \\ 9 & -14 \end{bmatrix}$$

$$B^T = \begin{bmatrix} -1 & -1 & 3 \\ 2 & -3 & 4 \end{bmatrix},\ A^T = \begin{bmatrix} 3 & -1 \\ -1 & 4 \\ 0 & 0 \end{bmatrix}$$

$$B^T A^T = \begin{bmatrix} -2 & -3 \\ 9 & -14 \end{bmatrix}$$

So $(AB)^T = B^T A^T$.

Ans: Both equal $\begin{bmatrix} -2 & -3 \\ 9 & -14 \end{bmatrix}$

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13. Find x if $\begin{bmatrix} 9 & -8 & -3 \\ -8 & -8 & -4x-3 \\ -3 & 1 & -7 \end{bmatrix}$ is symmetric. [1]

Symmetric means $a_{ij} = a_{ji}$.

$a_{23} = a_{32} \Rightarrow -4x - 3 = 1$

$x = -1$

Ans: $x = -1$

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14. Find a, b, c if $\begin{bmatrix} 0 & 6 & b \\ a & 0 & -8 \\ 1 & c & 0 \end{bmatrix}$ is skew-symmetric. [2]

Skew-symmetric means $a_{ij} = -a_{ji}$.

$a = -(6) = -6, \quad b = -(1) = -1, \quad c = -(-8) = 8$

Ans: $a = -6$, $b = -1$, $c = 8$

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15. Express $A = \begin{bmatrix} 6 & -6 & 2 \\ -4 & 4 & -5 \\ 3 & 6 & 1 \end{bmatrix}$ as the sum of a symmetric and a skew-symmetric matrix. [5]

$A^T = \begin{bmatrix} 6 & -4 & 3 \\ -6 & 4 & 6 \\ 2 & -5 & 1 \end{bmatrix}$

$A + A^T = \begin{bmatrix} 12 & -10 & 5 \\ -10 & 8 & 1 \\ 5 & 1 & 2 \end{bmatrix}$, $P = \frac{1}{2}(A + A^T) = \begin{bmatrix} 6 & -5 & \frac{5}{2} \\ -5 & 4 & \frac{1}{2} \\ \frac{5}{2} & \frac{1}{2} & 1 \end{bmatrix}$

$A - A^T = \begin{bmatrix} 0 & -2 & -1 \\ 2 & 0 & -11 \\ 1 & 11 & 0 \end{bmatrix}$, $Q = \frac{1}{2}(A - A^T) = \begin{bmatrix} 0 & -1 & -\frac{1}{2} \\ 1 & 0 & -\frac{11}{2} \\ \frac{1}{2} & \frac{11}{2} & 0 \end{bmatrix}$

$P^T = P$ and $Q^T = -Q$, and $P + Q = A$.

$A = \begin{bmatrix} 6 & -5 & \frac{5}{2} \\ -5 & 4 & \frac{1}{2} \\ \frac{5}{2} & \frac{1}{2} & 1 \end{bmatrix} + \begin{bmatrix} 0 & -1 & -\frac{1}{2} \\ 1 & 0 & -\frac{11}{2} \\ \frac{1}{2} & \frac{11}{2} & 0 \end{bmatrix}$

Ans: $\begin{bmatrix} 6 & -5 & \frac{5}{2} \\ -5 & 4 & \frac{1}{2} \\ \frac{5}{2} & \frac{1}{2} & 1 \end{bmatrix} + \begin{bmatrix} 0 & -1 & -\frac{1}{2} \\ 1 & 0 & -\frac{11}{2} \\ \frac{1}{2} & \frac{11}{2} & 0 \end{bmatrix}$

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16. Show that $A = \frac{1}{9}\begin{bmatrix} 8 & 1 & 4 \\ 4 & -4 & -7 \\ 1 & 8 & -4 \end{bmatrix}$ is an orthogonal matrix. [5]

$A^T = \frac{1}{9}\begin{bmatrix} 8 & 4 & 1 \\ 1 & -4 & 8 \\ 4 & -7 & -4 \end{bmatrix}$

$AA^T = \frac{1}{81}\begin{bmatrix} 81 & 0 & 0 \\ 0 & 81 & 0 \\ 0 & 0 & 81 \end{bmatrix}$

$= \frac{1}{81} \cdot 81I = I$

Similarly $A^T A = I$, so A is orthogonal.

Ans: $AA^T = A^T A = I$, **so A is orthogonal**

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17. Show that $A = \begin{bmatrix} 2 & 1 \\ -2 & -1 \end{bmatrix}$ is idempotent. [2]

$A^2 = \begin{bmatrix} 2 & 1 \\ -2 & -1 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ -2 & -1 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ -2 & -1 \end{bmatrix}$

$A^2 = A$, so A is idempotent.

Ans: $A^2 = A$

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18. Find the conjugate $\bar{A}$ of $A = \begin{bmatrix} 3 - i & 2 + 5i \\ 3 - i & -5 - 4i \end{bmatrix}$. [1]

Replace each element by its complex conjugate.

$\bar{A} = \begin{bmatrix} 3 + i & 2 - 5i \\ 3 + i & 4i - 5 \end{bmatrix}$

Ans: $\bar{A} = \begin{bmatrix} 3 + i & 2 - 5i \\ 3 + i & 4i - 5 \end{bmatrix}$

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19. For $A = \begin{bmatrix} 6 & -1 \\ 6 & -5 \end{bmatrix}$, find numbers p and q with $A^3 = pA + qI$, using the equation that A satisfies. [3]

$A^2 - (a+d)A + (ad-bc)I = O \Rightarrow A^2 = A + 24I$

$A^3 = A \cdot A^2 = A^2 + 24A = 1(A + 24I) + 24A$

$= 25A + 24I$

So $p = 25$, $q = 24$.

Ans: $p = 25$, $q = 24$

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20. Find the trace, the sum of the 2×2 principal minors and $|A|$ for $A = \begin{bmatrix} 4 & -1 & 3 \\ 3 & -1 & -1 \\ -3 & 3 & -2 \end{bmatrix}$, and write the cubic equation that A satisfies. [5]

$\operatorname{tr} A = 4 - 1 - 2 = 1$

$\begin{vmatrix} -1 & -1 \\ 3 & -2 \end{vmatrix} = 5, \quad \begin{vmatrix} 4 & 3 \\ -3 & -2 \end{vmatrix} = 1, \quad \begin{vmatrix} 4 & -1 \\ 3 & -1 \end{vmatrix} = -1$

Sum of the principal minors $= 5$.

$|A| = 29$

So $A^3 - A^2 + 5A - 29I = O$.

Ans: $A^3 - A^2 + 5A - 29I = O$
