EXPRESSWALA ACADEMY

Maths 1A: Algebra of Matrices

Question Paper, Set 04

Time: 1 hour 30 minutes | Maximum marks: 50

Name: _____ Class: _____ Date: _____

1. Answer all 20 questions. The marks for each question are shown in brackets.
2. Show your working for the 2, 3 and 5 mark questions.
3. The key sheet and worked solutions are at the end of this paper.

1. For $A = \begin{bmatrix} 10 & 6 \\ -5 & 0 \\ 3 & 8 \end{bmatrix}$, find the number of elements of A and $a_{21} + a_{11}$. [1 mark]

2. Write the matrix $A = [a_{ij}]_{3\times 2}$ whose elements are $a_{ij} = |i - 2j|$. [2 marks]

3. Name the type of the square matrix $\begin{bmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix}$ (the most special type that fits). [1 mark]

4. Find the trace of $\begin{bmatrix} -3 & -3 & -6 \\ 5 & 3 & 6 \\ 0 & 2 & -2 \end{bmatrix}$. [1 mark]

5. If $\begin{bmatrix} 3x - y & -1 \\ -1 & 2x + 5y \end{bmatrix} = \begin{bmatrix} -1 & -1 \\ -1 & 22 \end{bmatrix}$, find the values of x and y. [2 marks]

6. Find the matrix X if $3X + \begin{bmatrix} -1 & -5 & -6 \\ 0 & -5 & -2 \end{bmatrix} = \begin{bmatrix} -7 & -11 & -6 \\ 12 & -2 & 13 \end{bmatrix}$. [2 marks]

7. Find X and Y if $2X + Y = \begin{bmatrix} -6 & 2 \\ 4 & 10 \end{bmatrix}$ and $X + 2Y = \begin{bmatrix} -6 & 1 \\ -1 & 14 \end{bmatrix}$. [3 marks]

8. If $A = \begin{bmatrix} 3 & -4 \\ 2 & 0 \\ 1 & -2 \end{bmatrix}$ and $B = \begin{bmatrix} 4 \\ -4 \end{bmatrix}$, find AB. [2 marks]

9. If $A = \begin{bmatrix} 6 & 2 \\ 2 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 5 \\ 0 & 4 \end{bmatrix}$, find AB and BA. Is $AB = BA$? [3 marks]

10. If $A = \begin{bmatrix} -2 & 1 \\ 1 & -1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 4 \\ 0 & 0 \end{bmatrix}$, find $(A + B)^2$ and $A^2 + 2AB + B^2$. Are they equal? Why? [5 marks]

11. Find A^T for $A = \begin{bmatrix} 4 & 6 & -4 & -5 \end{bmatrix}$ and state its order. [1 mark]

12. If $A = \begin{bmatrix} -3 & 3 & 5 \\ -5 & -4 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & 4 & 6 \\ -5 & -2 & 2 \end{bmatrix}$, verify that $(3A - B)^T = 3A^T - B^T$. [3 marks]

13. Find x if $\begin{bmatrix} 3 & -3x+2 & 6 \\ 8 & 4 & -1 \\ 6 & -1 & 4 \end{bmatrix}$ is symmetric. [1 mark]

14. For $A = \begin{bmatrix} -2 & 6 & 2 \\ 0 & -5 & -1 \\ 1 & -3 & -4 \end{bmatrix}$, find $A - A^T$ and show that it is skew-symmetric. [2 marks]

15. Write $\begin{bmatrix} -4 & -6 & 2 \\ 5 & -5 & 3 \\ 3 & -1 & 7 \end{bmatrix}$ as $P + Q$ where P is symmetric and Q is skew-symmetric. [5 marks]

16. Find the positive number k for which $k\begin{bmatrix} -2 & -1 & -2 \\ -2 & 2 & 1 \\ -1 & -2 & 2 \end{bmatrix}$ is orthogonal, and check one row and two columns of your answer. [5 marks]

17. Show that $A = \begin{bmatrix} 3 & -1 \\ 6 & -2 \end{bmatrix}$ is idempotent. [2 marks]

18. Find the transposed conjugate A^θ of $A = \begin{bmatrix} 3-i & -4-i \\ 4-5i & 3i-1 \end{bmatrix}$. [1 mark]

19. Show that $A = \begin{bmatrix} 2 & -2 \\ 5 & 2 \end{bmatrix}$ satisfies $A^2 - (a+d)A + (ad-bc)I = O$, writing the equation with numbers. [3 marks]

20. Write the equation $A^3 - (\operatorname{tr} A)A^2 + (\ldots)A - |A|I = O$ satisfied by $A = \begin{bmatrix} 4 & 2 & 2 \\ -3 & 3 & -1 \\ -3 & 4 & -3 \end{bmatrix}$, finding every coefficient. [5 marks]

Key sheet

Q	Answer
1	6 elements; $a_{21} + a_{11} = 5$
2	$A = \begin{bmatrix} 1 & 3 \\ 0 & 2 \\ 1 & 1 \end{bmatrix}$
3	A scalar matrix
4	-2
5	$x = 1, y = 4$
6	$X = \begin{bmatrix} -2 & -2 & 0 \\ 4 & 1 & 5 \end{bmatrix}$
7	$X = \begin{bmatrix} -2 & 1 \\ 3 & 2 \end{bmatrix}, Y = \begin{bmatrix} -2 & 0 \\ -2 & 6 \end{bmatrix}$
8	$AB = \begin{bmatrix} 28 \\ 8 \\ 12 \end{bmatrix}$
9	$AB = \begin{bmatrix} 6 & 38 \\ 2 & 18 \end{bmatrix}, BA = \begin{bmatrix} 16 & 12 \\ 8 & 8 \end{bmatrix}$; $AB \neq BA$
10	$(A+B)^2 = \begin{bmatrix} 6 & -10 \\ -2 & 6 \end{bmatrix}, A^2 + 2AB + B^2 = \begin{bmatrix} 2 & -15 \\ -1 & 10 \end{bmatrix}$; not equal since $AB \neq BA$
11	$A^T = \begin{bmatrix} 4 \\ 6 \\ -4 \\ -5 \end{bmatrix}$, order 4×1
12	Both equal $\begin{bmatrix} -12 & -10 \\ 5 & -10 \\ 9 & 7 \end{bmatrix}$
13	$x = -2$
14	$A - A^T = \begin{bmatrix} 0 & 6 & 1 \\ -6 & 0 & 2 \\ -1 & -2 & 0 \end{bmatrix}$
15	$\begin{bmatrix} -4 & -\frac{1}{2} & \frac{5}{2} \\ -\frac{1}{2} & -5 & 1 \\ \frac{5}{2} & 1 & 7 \end{bmatrix} + \begin{bmatrix} 0 & -\frac{11}{2} & -\frac{1}{2} \\ \frac{11}{2} & 0 & 2 \\ \frac{1}{2} & -2 & 0 \end{bmatrix}$
16	$k = \frac{1}{3}$
17	$A^2 = A$
18	$A^\theta = \begin{bmatrix} 3+i & 4+5i \\ i-4 & -1-3i \end{bmatrix}$
19	$A^2 - 4A + 14I = O$
20	$A^3 - 4A^2 + 7A + 38I = O$

Solutions

1. For $A = \begin{bmatrix} 10 & 6 \\ -5 & 0 \\ 3 & 8 \end{bmatrix}$, find the number of elements of A and $a_{21} + a_{11}$. [1]

Order 3×2, so $3 \times 2 = 6$ elements.

$a_{21} + a_{11} = -5 + 10 = 5$

Ans: 6 elements; $a_{21} + a_{11} = 5$

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2. Write the matrix $A = [a_{ij}]_{3\times 2}$ whose elements are $a_{ij} = |i - 2j|$. [2]

$a_{ij} = |i - 2j|$

$a_{11} = 1$, $a_{12} = 3$

$a_{21} = 0$, $a_{22} = 2$

$a_{31} = 1$, $a_{32} = 1$

$A = \begin{bmatrix} 1 & 3 \\ 0 & 2 \\ 1 & 1 \end{bmatrix}$

Ans: $A = \begin{bmatrix} 1 & 3 \\ 0 & 2 \\ 1 & 1 \end{bmatrix}$

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3. Name the type of the square matrix $\begin{bmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix}$ (the most special type that fits). [1]

Here it is diagonal and all its diagonal elements are equal to -2.

So it is a scalar matrix.

Ans: A scalar matrix

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4. Find the trace of $\begin{bmatrix} -3 & -3 & -6 \\ 5 & 3 & 6 \\ 0 & 2 & -2 \end{bmatrix}$. [1]

$\text{tr}(A) = a_{11} + a_{22} + a_{33}$

$= -3 + 3 - 2 = -2$

Ans: -2

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5. If $\begin{bmatrix} 3x - y & -1 \\ -1 & 2x + 5y \end{bmatrix} = \begin{bmatrix} -1 & -1 \\ -1 & 22 \end{bmatrix}$, find the values of x and y. [2]

Equate corresponding elements:

$3x - y = -1, \quad 2x + 5y = 22$

Solving, $x = 1$, $y = 4$.

Ans: $x = 1$, $y = 4$

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6. Find the matrix X if $3X + \begin{bmatrix} -1 & -5 & -6 \\ 0 & -5 & -2 \end{bmatrix} = \begin{bmatrix} -7 & -11 & -6 \\ 12 & -2 & 13 \end{bmatrix}$. [2]

$3X = \begin{bmatrix} -7 & -11 & -6 \\ 12 & -2 & 13 \end{bmatrix} - \begin{bmatrix} -1 & -5 & -6 \\ 0 & -5 & -2 \end{bmatrix} = \begin{bmatrix} -6 & -6 & 0 \\ 12 & 3 & 15 \end{bmatrix}$

$X = \frac{1}{3}\begin{bmatrix} -6 & -6 & 0 \\ 12 & 3 & 15 \end{bmatrix} = \begin{bmatrix} -2 & -2 & 0 \\ 4 & 1 & 5 \end{bmatrix}$

Ans: $X = \begin{bmatrix} -2 & -2 & 0 \\ 4 & 1 & 5 \end{bmatrix}$

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7. Find X and Y if $2X + Y = \begin{bmatrix} -6 & 2 \\ 4 & 10 \end{bmatrix}$ and $X + 2Y = \begin{bmatrix} -6 & 1 \\ -1 & 14 \end{bmatrix}$. [3]

Multiply the first by 2 and subtract the second: $3X = 2\begin{bmatrix} -6 & 2 \\ 4 & 10 \end{bmatrix} - \begin{bmatrix} -6 & 1 \\ -1 & 14 \end{bmatrix} = \begin{bmatrix} -6 & 3 \\ 9 & 6 \end{bmatrix}$, so $X = \begin{bmatrix} -2 & 1 \\ 3 & 2 \end{bmatrix}$.

Then $Y = \begin{bmatrix} -6 & 2 \\ 4 & 10 \end{bmatrix} - 2X = \begin{bmatrix} -2 & 0 \\ -2 & 6 \end{bmatrix}$.

Ans: $X = \begin{bmatrix} -2 & 1 \\ 3 & 2 \end{bmatrix}$, $Y = \begin{bmatrix} -2 & 0 \\ -2 & 6 \end{bmatrix}$

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8. If $A = \begin{bmatrix} 3 & -4 \\ 2 & 0 \\ 1 & -2 \end{bmatrix}$ and $B = \begin{bmatrix} 4 \\ -4 \end{bmatrix}$, find AB. [2]

A is 3×2 and B is 2×1, so AB is 3×1.

$c_{11} = (3)(4) + (-4)(-4) = 28$

$c_{21} = (2)(4) + (0)(-4) = 8$

$c_{31} = (1)(4) + (-2)(-4) = 12$

$AB = \begin{bmatrix} 28 \\ 8 \\ 12 \end{bmatrix}$

Ans: $AB = \begin{bmatrix} 28 \\ 8 \\ 12 \end{bmatrix}$

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9. If $A = \begin{bmatrix} 6 & 2 \\ 2 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 5 \\ 0 & 4 \end{bmatrix}$, find AB and BA. Is $AB = BA$? [3]

$AB = \begin{bmatrix} 6 & 38 \\ 2 & 18 \end{bmatrix}$

$BA = \begin{bmatrix} 16 & 12 \\ 8 & 8 \end{bmatrix}$

The products differ, so $AB \neq BA$: matrix multiplication is not commutative.

Ans: $AB = \begin{bmatrix} 6 & 38 \\ 2 & 18 \end{bmatrix}$, $BA = \begin{bmatrix} 16 & 12 \\ 8 & 8 \end{bmatrix}$; $AB \neq BA$

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10. If $A = \begin{bmatrix} -2 & 1 \\ 1 & -1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 4 \\ 0 & 0 \end{bmatrix}$, find $(A+B)^2$ and $A^2 + 2AB + B^2$. Are they equal? Why? [5]

$A + B = \begin{bmatrix} -1 & 5 \\ 1 & -1 \end{bmatrix}$, $(A+B)^2 = \begin{bmatrix} 6 & -10 \\ -2 & 6 \end{bmatrix}$

$A^2 = \begin{bmatrix} 5 & -3 \\ -3 & 2 \end{bmatrix}$, $B^2 = \begin{bmatrix} 1 & 4 \\ 0 & 0 \end{bmatrix}$

$AB = \begin{bmatrix} -2 & -8 \\ 1 & 4 \end{bmatrix}$, $BA = \begin{bmatrix} 2 & -3 \\ 0 & 0 \end{bmatrix}$

$A^2 + 2AB + B^2 = \begin{bmatrix} 2 & -15 \\ -1 & 10 \end{bmatrix}$

They differ because $(A+B)^2 = A^2 + AB + BA + B^2$ and here $AB \neq BA$.

Ans: $(A+B)^2 = \begin{bmatrix} 6 & -10 \\ -2 & 6 \end{bmatrix}$, $A^2 + 2AB + B^2 = \begin{bmatrix} 2 & -15 \\ -1 & 10 \end{bmatrix}$; **not equal since** $AB \neq BA$

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11. Find A^T for $A = \begin{bmatrix} 4 & 6 & -4 & -5 \end{bmatrix}$ and state its order. [1]

Rows of A become columns of A^T.

$A^T = \begin{bmatrix} 4 \\ 6 \\ -4 \\ -5 \end{bmatrix}$

Order 4×1.

Ans: $A^T = \begin{bmatrix} 4 \\ 6 \\ -4 \\ -5 \end{bmatrix}$, **order** 4×1

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12. If $A = \begin{bmatrix} -3 & 3 & 5 \\ -5 & -4 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & 4 & 6 \\ -5 & -2 & 2 \end{bmatrix}$, verify that $(3A - B)^T = 3A^T - B^T$. [3]

$3A - B = \begin{bmatrix} -12 & 5 & 9 \\ -10 & -10 & 7 \end{bmatrix}$

$(3A-B)^T = \begin{bmatrix} -12 & -10 \\ 5 & -10 \\ 9 & 7 \end{bmatrix}$

$3A^T - B^T = \begin{bmatrix} -9 & -15 \\ 9 & -12 \\ 15 & 9 \end{bmatrix} - \begin{bmatrix} 3 & -5 \\ 4 & -2 \\ 6 & 2 \end{bmatrix} = \begin{bmatrix} -12 & -10 \\ 5 & -10 \\ 9 & 7 \end{bmatrix}$

The two are equal.

Ans: Both equal $\begin{bmatrix} -12 & -10 \\ 5 & -10 \\ 9 & 7 \end{bmatrix}$

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13. Find x if $\begin{bmatrix} 3 & -3x+2 & 6 \\ 8 & 4 & -1 \\ 6 & -1 & 4 \end{bmatrix}$ is symmetric. [1]

Symmetric means $a_{ij} = a_{ji}$.

$a_{12} = a_{21} \Rightarrow -3x + 2 = 8$

$x = -2$

Ans: $x = -2$

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14. For $A = \begin{bmatrix} -2 & 6 & 2 \\ 0 & -5 & -1 \\ 1 & -3 & -4 \end{bmatrix}$, find $A - A^T$ and show that it is skew-symmetric. [2]

$A^T = \begin{bmatrix} -2 & 0 & 1 \\ 6 & -5 & -3 \\ 2 & -1 & -4 \end{bmatrix}$

$A - A^T = \begin{bmatrix} 0 & 6 & 1 \\ -6 & 0 & 2 \\ -1 & -2 & 0 \end{bmatrix}$

Its diagonal is zero and $a_{ij} = -a_{ji}$, so it is skew-symmetric.

Ans: $A - A^T = \begin{bmatrix} 0 & 6 & 1 \\ -6 & 0 & 2 \\ -1 & -2 & 0 \end{bmatrix}$

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15. Write $\begin{bmatrix} -4 & -6 & 2 \\ 5 & -5 & 3 \\ 3 & -1 & 7 \end{bmatrix}$ as $P + Q$ where P is symmetric and Q is skew-symmetric. [5]

$A^T = \begin{bmatrix} -4 & 5 & 3 \\ -6 & -5 & -1 \\ 2 & 3 & 7 \end{bmatrix}$

$A + A^T = \begin{bmatrix} -8 & -1 & 5 \\ -1 & -10 & 2 \\ 5 & 2 & 14 \end{bmatrix}$, $\quad P = \frac{1}{2}(A + A^T) = \begin{bmatrix} -4 & -\frac{1}{2} & \frac{5}{2} \\ -\frac{1}{2} & -5 & 1 \\ \frac{5}{2} & 1 & 7 \end{bmatrix}$

$A - A^T = \begin{bmatrix} 0 & -11 & -1 \\ 11 & 0 & 4 \\ 1 & -4 & 0 \end{bmatrix}$, $\quad Q = \frac{1}{2}(A - A^T) = \begin{bmatrix} 0 & -\frac{11}{2} & -\frac{1}{2} \\ \frac{11}{2} & 0 & 2 \\ \frac{1}{2} & -2 & 0 \end{bmatrix}$

$P^T = P$ and $Q^T = -Q$, and $P + Q = A$.

$A = \begin{bmatrix} -4 & -\frac{1}{2} & \frac{5}{2} \\ -\frac{1}{2} & -5 & 1 \\ \frac{5}{2} & 1 & 7 \end{bmatrix} + \begin{bmatrix} 0 & -\frac{11}{2} & -\frac{1}{2} \\ \frac{11}{2} & 0 & 2 \\ \frac{1}{2} & -2 & 0 \end{bmatrix}$

Ans: $\begin{bmatrix} -4 & -\frac{1}{2} & \frac{5}{2} \\ -\frac{1}{2} & -5 & 1 \\ \frac{5}{2} & 1 & 7 \end{bmatrix} + \begin{bmatrix} 0 & -\frac{11}{2} & -\frac{1}{2} \\ \frac{11}{2} & 0 & 2 \\ \frac{1}{2} & -2 & 0 \end{bmatrix}$

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16. Find the positive number k for which $k \begin{bmatrix} -2 & -1 & -2 \\ -2 & 2 & 1 \\ -1 & -2 & 2 \end{bmatrix}$ is orthogonal, and check one row and two columns of your answer. [5]

Let $A = kM$. Then $AA^T = k^2 MM^T$.

$MM^T = \begin{bmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{bmatrix} = 9I$

$AA^T = I \Rightarrow 9k^2 = 1 \Rightarrow k = \frac{1}{3}$

Row 1: $\frac{1}{9}((-2)^2 + (-1)^2 + (-2)^2) = 1$.

Columns 1, 2: $\frac{1}{9}((-2)(-1) + (-2)(2) + (-1)(-2)) = 0$.

Ans: $k = \frac{1}{3}$

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17. Show that $A = \begin{bmatrix} 3 & -1 \\ 6 & -2 \end{bmatrix}$ is idempotent. [2]

$A^2 = \begin{bmatrix} 3 & -1 \\ 6 & -2 \end{bmatrix}\begin{bmatrix} 3 & -1 \\ 6 & -2 \end{bmatrix} = \begin{bmatrix} 3 & -1 \\ 6 & -2 \end{bmatrix}$

$A^2 = A$, so A is idempotent.

Ans: $A^2 = A$

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18. Find the transposed conjugate A^θ of $A = \begin{bmatrix} 3-i & -4-i \\ 4-5i & 3i-1 \end{bmatrix}$. [1]

$\bar{A} = \begin{bmatrix} 3+i & i-4 \\ 4+5i & -1-3i \end{bmatrix}$

$A^\theta = (\bar{A})^T = \begin{bmatrix} 3+i & 4+5i \\ i-4 & -1-3i \end{bmatrix}$

Ans: $A^\theta = \begin{bmatrix} 3+i & 4+5i \\ i-4 & -1-3i \end{bmatrix}$

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19. Show that $A = \begin{bmatrix} 2 & -2 \\ 5 & 2 \end{bmatrix}$ satisfies $A^2 - (a+d)A + (ad-bc)I = O$, writing the equation with numbers. [3]

$a + d = 4, \quad ad - bc = 14$

$A^2 = \begin{bmatrix} -6 & -8 \\ 20 & -6 \end{bmatrix}$

$A^2 - (4)A + (14)I = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

$A^2 - 4A + 14I = O$

Ans: $A^2 - 4A + 14I = O$

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20. Write the equation $A^3 - (\operatorname{tr} A)A^2 + (\ldots)A - |A|I = O$ satisfied by $A = \begin{bmatrix} 4 & 2 & 2 \\ -3 & 3 & -1 \\ -3 & 4 & -3 \end{bmatrix}$, finding every coefficient. [5]

$\operatorname{tr} A = 4 + 3 - 3 = 4$

$\begin{vmatrix} 3 & -1 \\ 4 & -3 \end{vmatrix} = -5, \quad \begin{vmatrix} 4 & 2 \\ -3 & -3 \end{vmatrix} = -6, \quad \begin{vmatrix} 4 & 2 \\ -3 & 3 \end{vmatrix} = 18$

Sum of the principal minors $= 7$.

$|A| = -38$

So $A^3 - 4A^2 + 7A + 38I = O$.

Ans: $A^3 - 4A^2 + 7A + 38I = O$
