EXPRESSWALA ACADEMY

# Maths 1A: Algebra of Matrices

## Question Paper, Set 03

Time: 1 hour 30 minutes | Maximum marks: 50

Name: _____ Class: _____ Date: _____

1. Answer all 20 questions. The marks for each question are shown in brackets.
2. Show your working for the 2, 3 and 5 mark questions.
3. The key sheet and worked solutions are at the end of this paper.

**1.** Write the order of $A = \begin{bmatrix} 7 & 7 & -6 \\ -3 & 1 & -9 \end{bmatrix}$ and the element $a_{22}$. [1 mark]

**2.** Construct the $3 \times 3$ matrix $A = [a_{ij}]$ with $a_{ij} = |i - 2j|$. [2 marks]

**3.** Name the type of the square matrix $\begin{bmatrix} -2 & 0 \\ 0 & 7 \end{bmatrix}$ (the most special type that fits). [1 mark]

**4.** If $A = \begin{bmatrix} -5 & -4 \\ 2 & -3 \end{bmatrix}$, find $\operatorname{tr}(5I_5) + \operatorname{tr}(A^T)$. [1 mark]

**5.** Find $x$ and $y$ if $\begin{bmatrix} 2x - 3y & -1 \\ 8 & x + 2y \end{bmatrix} = \begin{bmatrix} -13 & -1 \\ 8 & -3 \end{bmatrix}$. [2 marks]

**6.** If $A = \begin{bmatrix} -4 & -1 & 2 \\ -4 & 6 & -5 \end{bmatrix}$ and $B = \begin{bmatrix} 5 & 7 & -5 \\ 5 & 2 & 7 \end{bmatrix}$, find $2A - 2B$. [2 marks]

**7.** Find the matrices $X$ and $Y$ if $X + Y = \begin{bmatrix} 5 & 8 \\ 4 & 6 \end{bmatrix}$ and $X - Y = \begin{bmatrix} 3 & -4 \\ 4 & -6 \end{bmatrix}$. [3 marks]

**8.** If $A = \begin{bmatrix} -3 & 2 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 4 \\ 0 & 5 \\ 4 & 1 \end{bmatrix}$, find $AB$. [2 marks]

**9.** If $A = \begin{bmatrix} 1 & -2 & 0 \\ 2 & 2 & -2 \end{bmatrix}$ and $B = \begin{bmatrix} 4 & -3 \\ 2 & 3 \\ 1 & 4 \end{bmatrix}$, find $AB$ and $BA$. [3 marks]

**10.** If $A = \begin{bmatrix} -2 & -1 & 1 \\ -1 & 1 & -1 \\ 1 & 0 & 0 \end{bmatrix}$, find $A^2 - 5A + 1I$. [5 marks]

**11.** Write the transpose of $A = \begin{bmatrix} 5 & 5 & 0 & -9 \end{bmatrix}$ and its order. [1 mark]

**12.** If $A = \begin{bmatrix} -3 & 2 & -2 \\ -3 & -1 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} -3 & -1 \\ -3 & 0 \\ 2 & 4 \end{bmatrix}$, verify that $(AB)^T = B^T A^T$. [3 marks]

**13.** Find $x$ if $\begin{bmatrix} 4 & -7 & -x - 2 \\ -7 & 7 & -8 \\ 8 & -8 & 5 \end{bmatrix}$ is symmetric. [1 mark]

**14.** Find $a$, $b$, $c$ if $\begin{bmatrix} 0 & -8 & b \\ a & 0 & 7 \\ 4 & c & 0 \end{bmatrix}$ is skew-symmetric. [2 marks]

**15.** Express $A = \begin{bmatrix} 3 & 5 & 0 \\ 3 & 1 & 0 \\ 6 & -6 & -3 \end{bmatrix}$ as the sum of a symmetric and a skew-symmetric matrix. [5 marks]

**16.** Show that $A = \frac{1}{3}\begin{bmatrix} 1 & 2 & -2 \\ 2 & 1 & 2 \\ 2 & -2 & -1 \end{bmatrix}$ is an orthogonal matrix. [5 marks]

**17.** Show that $A = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}$ is nilpotent and find its index. [2 marks]

**18.** Find the conjugate $\bar{A}$ of $A = \begin{bmatrix} 5-2i & 1+7i \\ 2+7i & -5-5i \end{bmatrix}$. [1 mark]

**19.** For $A = \begin{bmatrix} 1 & 4 \\ -4 & 3 \end{bmatrix}$, find numbers $p$ and $q$ with $A^3 = pA + qI$, using the equation that $A$ satisfies. [3 marks]

**20.** Find the trace, the sum of the $2 \times 2$ principal minors and $|A|$ for $A = \begin{bmatrix} 4 & -3 & -3 \\ -1 & 0 & 4 \\ -2 & -2 & 0 \end{bmatrix}$, and write the cubic equation that $A$ satisfies. [5 marks]

## Key sheet

| Q | Answer |
|---|---|
| 1 | Order $2 \times 3$; $a_{22} = 1$ |
| 2 | $A = \begin{bmatrix} 1 & 3 & 5 \\ 0 & 2 & 4 \\ 1 & 1 & 3 \end{bmatrix}$ |
| 3 | A diagonal matrix (but not a scalar matrix) |
| 4 | $17$ |
| 5 | $x = -5$, $y = 1$ |
| 6 | $\begin{bmatrix} -18 & -16 & 14 \\ -18 & 8 & -24 \end{bmatrix}$ |
| 7 | $X = \begin{bmatrix} 4 & 2 \\ 4 & 0 \end{bmatrix}$, $Y = \begin{bmatrix} 1 & 6 \\ 0 & 6 \end{bmatrix}$ |
| 8 | $AB = \begin{bmatrix} 16 & 2 \end{bmatrix}$ |
| 9 | $AB = \begin{bmatrix} 0 & -9 \\ 10 & -8 \end{bmatrix}$, $BA = \begin{bmatrix} -2 & -14 & 6 \\ 8 & 2 & -6 \\ 9 & 6 & -8 \end{bmatrix}$ |
| 10 | $\begin{bmatrix} 17 & 6 & -6 \\ 5 & -2 & 3 \\ -7 & -1 & 2 \end{bmatrix}$ |
| 11 | $A^T = \begin{bmatrix} 5 \\ 5 \\ 0 \\ -9 \end{bmatrix}$, order $4 \times 1$ |
| 12 | Both equal $\begin{bmatrix} -1 & 16 \\ -5 & 11 \end{bmatrix}$ |
| 13 | $x = -10$ |
| 14 | $a = 8$, $b = -4$, $c = -7$ |
| 15 | $\begin{bmatrix} 3 & 4 & 3 \\ 4 & 1 & -3 \\ 3 & -3 & -3 \end{bmatrix} + \begin{bmatrix} 0 & 1 & -3 \\ -1 & 0 & 3 \\ 3 & -3 & 0 \end{bmatrix}$ |
| 16 | $AA^T = A^TA = I$, so $A$ is orthogonal |
| 17 | Nilpotent of index $2$ |
| 18 | $\bar{A} = \begin{bmatrix} 5 + 2i & 1 - 7i \\ 2 - 7i & 5i - 5 \end{bmatrix}$ |
| 19 | $p = -3$, $q = -76$ |
| 20 | $A^3 - 4A^2 - A - 50I = O$ |

## Solutions

**1.** Write the order of $A = \begin{bmatrix} 7 & 7 & -6 \\ -3 & 1 & -9 \end{bmatrix}$ and the element $a_{22}$. [1]

$A$ has 2 rows and 3 columns, so its order is $2 \times 3$.

$a_{22}$ is in row 2, column 2: $a_{22} = 1$.

**Ans: Order $2 \times 3$; $a_{22} = 1$**

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**2.** Construct the $3 \times 3$ matrix $A = [a_{ij}]$ with $a_{ij} = |i - 2j|$. [2]

$a_{ij} = |i - 2j|$

$a_{11} = 1$, $a_{12} = 3$, $a_{13} = 5$

$a_{21} = 0$, $a_{22} = 2$, $a_{23} = 4$

$a_{31} = 1$, $a_{32} = 1$, $a_{33} = 3$

$A = \begin{bmatrix} 1 & 3 & 5 \\ 0 & 2 & 4 \\ 1 & 1 & 3 \end{bmatrix}$

**Ans:** $A = \begin{bmatrix} 1 & 3 & 5 \\ 0 & 2 & 4 \\ 1 & 1 & 3 \end{bmatrix}$

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**3.** Name the type of the square matrix $\begin{bmatrix} -2 & 0 \\ 0 & 7 \end{bmatrix}$ (the most special type that fits). [1]

Here every element outside the principal diagonal is 0 and the diagonal elements are not all equal. So it is a diagonal matrix (but not a scalar matrix).

**Ans: A diagonal matrix (but not a scalar matrix)**

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**4.** If $A = \begin{bmatrix} -5 & -4 \\ 2 & -3 \end{bmatrix}$, find $\operatorname{tr}(5I_5) + \operatorname{tr}(A^T)$. [1]

$\operatorname{tr}(5I_5) = 5 \times 5 = 25$

$\operatorname{tr}(A^T) = \operatorname{tr}(A) = -8$

Sum $= 17$

**Ans:** $17$

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**5.** Find $x$ and $y$ if $\begin{bmatrix} 2x - 3y & -1 \\ 8 & x + 2y \end{bmatrix} = \begin{bmatrix} -13 & -1 \\ 8 & -3 \end{bmatrix}$. [2]

Equate corresponding elements:

$2x - 3y = -13, \quad x + 2y = -3$

Solving, $x = -5$, $y = 1$.

**Ans:** $x = -5$, $y = 1$

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**6.** If $A = \begin{bmatrix} -4 & -1 & 2 \\ -4 & 6 & -5 \end{bmatrix}$ and $B = \begin{bmatrix} 5 & 7 & -5 \\ 5 & 2 & 7 \end{bmatrix}$, find $2A - 2B$. [2]

$2A = \begin{bmatrix} -8 & -2 & 4 \\ -8 & 12 & -10 \end{bmatrix}$, $2B = \begin{bmatrix} 10 & 14 & -10 \\ 10 & 4 & 14 \end{bmatrix}$

$2A - 2B = \begin{bmatrix} -18 & -16 & 14 \\ -18 & 8 & -24 \end{bmatrix}$

**Ans:** $\begin{bmatrix} -18 & -16 & 14 \\ -18 & 8 & -24 \end{bmatrix}$

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**7.** Find the matrices $X$ and $Y$ if $X + Y = \begin{bmatrix} 5 & 8 \\ 4 & 6 \end{bmatrix}$ and $X - Y = \begin{bmatrix} 3 & -4 \\ 4 & -6 \end{bmatrix}$. [3]

Adding, $2X = \begin{bmatrix} 8 & 4 \\ 8 & 0 \end{bmatrix}$, so $X = \begin{bmatrix} 4 & 2 \\ 4 & 0 \end{bmatrix}$.

Subtracting, $2Y = \begin{bmatrix} 2 & 12 \\ 0 & 12 \end{bmatrix}$, so $Y = \begin{bmatrix} 1 & 6 \\ 0 & 6 \end{bmatrix}$.

**Ans:** $X = \begin{bmatrix} 4 & 2 \\ 4 & 0 \end{bmatrix}$, $Y = \begin{bmatrix} 1 & 6 \\ 0 & 6 \end{bmatrix}$

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**8.** If $A = \begin{bmatrix} -3 & 2 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 4 \\ 0 & 5 \\ 4 & 1 \end{bmatrix}$, find $AB$. [2]

$A$ is $1 \times 3$ and $B$ is $3 \times 2$, so $AB$ is $1 \times 2$.

$c_{11} = (-3)(0) + (2)(0) + (4)(4) = 16$

$c_{12} = (-3)(4) + (2)(5) + (4)(1) = 2$

$AB = \begin{bmatrix} 16 & 2 \end{bmatrix}$

**Ans:** $AB = \begin{bmatrix} 16 & 2 \end{bmatrix}$

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**9.** If $A = \begin{bmatrix} 1 & -2 & 0 \\ 2 & 2 & -2 \end{bmatrix}$ and $B = \begin{bmatrix} 4 & -3 \\ 2 & 3 \\ 1 & 4 \end{bmatrix}$, find $AB$ and $BA$. [3]

$AB$ is $2 \times 2$ and $BA$ is $3 \times 3$.

$AB = \begin{bmatrix} 0 & -9 \\ 10 & -8 \end{bmatrix}$

$BA = \begin{bmatrix} -2 & -14 & 6 \\ 8 & 2 & -6 \\ 9 & 6 & -8 \end{bmatrix}$

The two products are not even of the same order.

**Ans:** $AB = \begin{bmatrix} 0 & -9 \\ 10 & -8 \end{bmatrix}$, $BA = \begin{bmatrix} -2 & -14 & 6 \\ 8 & 2 & -6 \\ 9 & 6 & -8 \end{bmatrix}$

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**10.** If $A = \begin{bmatrix} -2 & -1 & 1 \\ -1 & 1 & -1 \\ 1 & 0 & 0 \end{bmatrix}$, find $A^2 - 5A + 1I$. [5]

$A^2 = A \cdot A = \begin{bmatrix} 6 & 1 & -1 \\ 0 & 2 & -2 \\ -2 & -1 & 1 \end{bmatrix}$

$A^2 - 5A + 1I = \begin{bmatrix} 6 & 1 & -1 \\ 0 & 2 & -2 \\ -2 & -1 & 1 \end{bmatrix} - \begin{bmatrix} -10 & -5 & 5 \\ -5 & 5 & -5 \\ 5 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$= \begin{bmatrix} 17 & 6 & -6 \\ 5 & -2 & 3 \\ -7 & -1 & 2 \end{bmatrix}$

**Ans:** $\begin{bmatrix} 17 & 6 & -6 \\ 5 & -2 & 3 \\ -7 & -1 & 2 \end{bmatrix}$

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**11.** Write the transpose of $A = \begin{bmatrix} 5 & 5 & 0 & -9 \end{bmatrix}$ and its order. [1]

Rows of $A$ become columns of $A^T$.

$A^T = \begin{bmatrix} 5 \\ 5 \\ 0 \\ -9 \end{bmatrix}$

Order $4 \times 1$.

**Ans:** $A^T = \begin{bmatrix} 5 \\ 5 \\ 0 \\ -9 \end{bmatrix}$, **order** $4 \times 1$

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**12.** If $A = \begin{bmatrix} -3 & 2 & -2 \\ -3 & -1 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} -3 & -1 \\ -3 & 0 \\ 2 & 4 \end{bmatrix}$, verify that $(AB)^T = B^T A^T$. [3]

$AB = \begin{bmatrix} -1 & -5 \\ 16 & 11 \end{bmatrix}$

$(AB)^T = \begin{bmatrix} -1 & 16 \\ -5 & 11 \end{bmatrix}$

$B^T = \begin{bmatrix} -3 & -3 & 2 \\ -1 & 0 & 4 \end{bmatrix}$, $A^T = \begin{bmatrix} -3 & -3 \\ 2 & -1 \\ -2 & 2 \end{bmatrix}$

$B^T A^T = \begin{bmatrix} -1 & 16 \\ -5 & 11 \end{bmatrix}$

So $(AB)^T = B^T A^T$.

**Ans: Both equal** $\begin{bmatrix} -1 & 16 \\ -5 & 11 \end{bmatrix}$

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**13.** Find $x$ if $\begin{bmatrix} 4 & -7 & -x-2 \\ -7 & 7 & -8 \\ 8 & -8 & 5 \end{bmatrix}$ is symmetric. [1]

Symmetric means $a_{ij} = a_{ji}$.

$a_{13} = a_{31} \Rightarrow -x - 2 = 8$

$x = -10$

**Ans:** $x = -10$

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**14.** Find $a$, $b$, $c$ if $\begin{bmatrix} 0 & -8 & b \\ a & 0 & 7 \\ 4 & c & 0 \end{bmatrix}$ is skew-symmetric. [2]

Skew-symmetric means $a_{ij} = -a_{ji}$.

$a = -(-8) = 8, \quad b = -(4) = -4, \quad c = -(7) = -7$

**Ans:** $a = 8$, $b = -4$, $c = -7$

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**15.** Express $A = \begin{bmatrix} 3 & 5 & 0 \\ 3 & 1 & 0 \\ 6 & -6 & -3 \end{bmatrix}$ as the sum of a symmetric and a skew-symmetric matrix. [5]

$A^T = \begin{bmatrix} 3 & 3 & 6 \\ 5 & 1 & -6 \\ 0 & 0 & -3 \end{bmatrix}$

$A + A^T = \begin{bmatrix} 6 & 8 & 6 \\ 8 & 2 & -6 \\ 6 & -6 & -6 \end{bmatrix}, \quad P = \frac{1}{2}(A + A^T) = \begin{bmatrix} 3 & 4 & 3 \\ 4 & 1 & -3 \\ 3 & -3 & -3 \end{bmatrix}$

$A - A^T = \begin{bmatrix} 0 & 2 & -6 \\ -2 & 0 & 6 \\ 6 & -6 & 0 \end{bmatrix}, \quad Q = \frac{1}{2}(A - A^T) = \begin{bmatrix} 0 & 1 & -3 \\ -1 & 0 & 3 \\ 3 & -3 & 0 \end{bmatrix}$

$P^T = P$ and $Q^T = -Q$, and $P + Q = A$.

$A = \begin{bmatrix} 3 & 4 & 3 \\ 4 & 1 & -3 \\ 3 & -3 & -3 \end{bmatrix} + \begin{bmatrix} 0 & 1 & -3 \\ -1 & 0 & 3 \\ 3 & -3 & 0 \end{bmatrix}$

**Ans:** $\begin{bmatrix} 3 & 4 & 3 \\ 4 & 1 & -3 \\ 3 & -3 & -3 \end{bmatrix} + \begin{bmatrix} 0 & 1 & -3 \\ -1 & 0 & 3 \\ 3 & -3 & 0 \end{bmatrix}$

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**16.** Show that $A = \frac{1}{3}\begin{bmatrix} 1 & 2 & -2 \\ 2 & 1 & 2 \\ 2 & -2 & -1 \end{bmatrix}$ is an orthogonal matrix. [5]

$A^T = \frac{1}{3}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ -2 & 2 & -1 \end{bmatrix}$

$AA^T = \frac{1}{9}\begin{bmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{bmatrix}$

$= \frac{1}{9} \cdot 9I = I$

Similarly $A^T A = I$, so $A$ is orthogonal.

**Ans:** $AA^T = A^T A = I$, **so $A$ is orthogonal**

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**17.** Show that $A = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}$ is nilpotent and find its index. [2]

$A \neq O, \quad A^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = O$

So $A$ is nilpotent of index $2$.

**Ans: Nilpotent of index $2$**

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**18.** Find the conjugate $\bar{A}$ of $A = \begin{bmatrix} 5-2i & 1+7i \\ 2+7i & -5-5i \end{bmatrix}$. [1]

Replace each element by its complex conjugate.

$\bar{A} = \begin{bmatrix} 5+2i & 1-7i \\ 2-7i & 5i-5 \end{bmatrix}$

**Ans:** $\bar{A} = \begin{bmatrix} 5+2i & 1-7i \\ 2-7i & 5i-5 \end{bmatrix}$

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**19.** For $A = \begin{bmatrix} 1 & 4 \\ -4 & 3 \end{bmatrix}$, find numbers $p$ and $q$ with $A^3 = pA + qI$, using the equation that $A$ satisfies. [3]

$A^2 - (a+d)A + (ad-bc)I = O \Rightarrow A^2 = 4A - 19I$

$A^3 = A \cdot A^2 = 4A^2 - 19A = 4(4A - 19I) - 19A$

$= -3A - 76I$

So $p = -3$, $q = -76$.

**Ans:** $p = -3$, $q = -76$

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**20.** Find the trace, the sum of the $2 \times 2$ principal minors and $|A|$ for $A = \begin{bmatrix} 4 & -3 & -3 \\ -1 & 0 & 4 \\ -2 & -2 & 0 \end{bmatrix}$, and write the cubic equation that $A$ satisfies. [5]

$\operatorname{tr} A = 4 + 0 + 0 = 4$

$\begin{vmatrix} 0 & 4 \\ -2 & 0 \end{vmatrix} = 8, \quad \begin{vmatrix} 4 & -3 \\ -2 & 0 \end{vmatrix} = -6, \quad \begin{vmatrix} 4 & -3 \\ -1 & 0 \end{vmatrix} = -3$

Sum of the principal minors $= -1$.

$|A| = 50$

So $A^3 - 4A^2 - A - 50I = O$.

**Ans:** $A^3 - 4A^2 - A - 50I = O$

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