EXPRESSWALA ACADEMY

Maths 1A: Algebra of Matrices

Question Paper, Set 02

Time: 1 hour 30 minutes | Maximum marks: 50

Name: _____ Class: _____ Date: _____

1. Answer all 20 questions. The marks for each question are shown in brackets.
2. Show your working for the 2, 3 and 5 mark questions.
3. The key sheet and worked solutions are at the end of this paper.

1. For $A = \begin{bmatrix} 8 & 3 \\ 8 & -3 \\ -5 & -4 \\ -9 & -4 \end{bmatrix}$, find the number of elements of A and $a_{42} + a_{31}$. [1 mark]

2. Write the matrix $A = [a_{ij}]_{2\times 2}$ whose elements are $a_{ij} = \frac{(i+j)^2}{2}$. [2 marks]

3. Name the type of the square matrix $\begin{bmatrix} -3 & 0 & 0 \\ -2 & 3 & 0 \\ -2 & -4 & -6 \end{bmatrix}$ (the most special type that fits). [1 mark]

4. If $\operatorname{tr}(A) = 8$ and $\operatorname{tr}(B) = -8$, find $\operatorname{tr}(4A - 4B)$. [1 mark]

5. If $\begin{bmatrix} 3x - y & -2 \\ 6 & 3x + 2y \end{bmatrix} = \begin{bmatrix} 11 & -2 \\ 6 & 23 \end{bmatrix}$, find the values of x and y. [2 marks]

6. Find the matrix X if $3X + \begin{bmatrix} 2 & -6 & -5 \\ 0 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 8 & -12 & 10 \\ -3 & 16 & 8 \end{bmatrix}$. [2 marks]

7. Find X and Y if $2X + Y = \begin{bmatrix} 11 & 6 \\ 9 & 18 \end{bmatrix}$ and $X + 2Y = \begin{bmatrix} 7 & 6 \\ 12 & 18 \end{bmatrix}$. [3 marks]

8. If $A = \begin{bmatrix} -1 & 4 & 3 \\ -4 & -1 & -3 \end{bmatrix}$ and $B = \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix}$, find AB. [2 marks]

9. If $A = \begin{bmatrix} -2 & 4 \\ -4 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & 5 \\ -4 & -2 \end{bmatrix}$, find AB and BA. Is $AB = BA$? [3 marks]

10. If $A = \begin{bmatrix} 3 & 1 \\ 1 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 4 & 4 \\ 1 & 0 \end{bmatrix}$, find $(A + B)^2$ and $A^2 + 2AB + B^2$. Are they equal? Why? [5 marks]

11. Find A^T for $A = \begin{bmatrix} 2 & 9 & -6 & -9 \end{bmatrix}$ and state its order. [1 mark]

12. If $A = \begin{bmatrix} -5 & -2 & -4 \\ -2 & -5 & -5 \end{bmatrix}$ and $B = \begin{bmatrix} -4 & -3 & -1 \\ 5 & -4 & -2 \end{bmatrix}$, verify that $(3A - B)^T = 3A^T - B^T$. [3 marks]

13. Find x if $\begin{bmatrix} 7 & 2 & -3x + 9 \\ 2 & 2 & -9 \\ -6 & -9 & -8 \end{bmatrix}$ is symmetric. [1 mark]

14. For $A = \begin{bmatrix} 7 & -1 & 5 \\ 5 & 3 & 4 \\ 6 & -3 & 5 \end{bmatrix}$, find $A - A^T$ and show that it is skew-symmetric. [2 marks]

15. Write $\begin{bmatrix} 7 & 6 & 3 \\ -3 & -3 & 1 \\ 3 & -6 & -4 \end{bmatrix}$ as $P + Q$ where P is symmetric and Q is skew-symmetric. [5 marks]

16. Find the positive number k for which $k \begin{bmatrix} 2 & -6 & -3 \\ -3 & 2 & -6 \\ -6 & -3 & 2 \end{bmatrix}$ is orthogonal, and check one row and two columns of your answer. [5 marks]

17. Show that $A = \begin{bmatrix} 4 & -5 \\ 3 & -4 \end{bmatrix}$ is involutory. [2 marks]

18. Find the transposed conjugate A^θ of $A = \begin{bmatrix} 7i - 2 & -4 - 3i \\ -2 - 2i & 7 + 3i \end{bmatrix}$. [1 mark]

19. Show that $A = \begin{bmatrix} -1 & 6 \\ 3 & -2 \end{bmatrix}$ satisfies $A^2 - (a + d)A + (ad - bc)I = O$, writing the equation with numbers. [3 marks]

20. Write the equation $A^3 - (\operatorname{tr} A)A^2 + (\ldots)A - |A|I = O$ satisfied by $A = \begin{bmatrix} -3 & -2 & -3 \\ 4 & 4 & 0 \\ 0 & 1 & 0 \end{bmatrix}$, finding every coefficient. [5 marks]

Key sheet

Q	Answer
1	8 elements; $a_{42}+a_{31}=-9$
2	$A=\begin{bmatrix}2 & \frac{9}{2}\\ \frac{9}{2} & 8\end{bmatrix}$
3	A lower triangular matrix
4	64
5	$x=5,\ y=4$
6	$X=\begin{bmatrix}2 & -2 & 5\\ -1 & 5 & 3\end{bmatrix}$
7	$X=\begin{bmatrix}5 & 2\\ 2 & 6\end{bmatrix},\ Y=\begin{bmatrix}1 & 2\\ 5 & 6\end{bmatrix}$
8	$AB=\begin{bmatrix}-3\\ -8\end{bmatrix}$
9	$AB=\begin{bmatrix}-14 & -18\\ -4 & -24\end{bmatrix},\ BA=\begin{bmatrix}-18 & 6\\ 16 & -20\end{bmatrix}$; $AB\neq BA$
10	$(A+B)^2=\begin{bmatrix}59 & 40\\ 16 & 11\end{bmatrix},\ A^2+2AB+B^2=\begin{bmatrix}56 & 44\\ 18 & 14\end{bmatrix}$; not equal since $AB\neq BA$
11	$A^T=\begin{bmatrix}2\\ 9\\ -6\\ -9\end{bmatrix}$, order 4×1
12	Both equal $\begin{bmatrix}-11 & -11\\ -3 & -11\\ -11 & -13\end{bmatrix}$
13	$x=5$
14	$A-A^T=\begin{bmatrix}0 & -6 & -1\\ 6 & 0 & 7\\ 1 & -7 & 0\end{bmatrix}$
15	$\begin{bmatrix}7 & \frac{3}{2} & 3\\ \frac{3}{2} & -3 & -\frac{5}{2}\\ 3 & -\frac{5}{2} & -4\end{bmatrix}+\begin{bmatrix}0 & \frac{9}{2} & 0\\ -\frac{9}{2} & 0 & \frac{7}{2}\\ 0 & -\frac{7}{2} & 0\end{bmatrix}$
16	$k=\frac{1}{7}$
17	$A^2=I$
18	$A^\theta=\begin{bmatrix}-2-7i & 2i-2\\ 3i-4 & 7-3i\end{bmatrix}$
19	$A^2+3A-16I=O$
20	$A^3-A^2-4A+12I=O$

Solutions

1. For $A = \begin{bmatrix} 8 & 3 \\ 8 & -3 \\ -5 & -4 \\ -9 & -4 \end{bmatrix}$, find the number of elements of A and $a_{42} + a_{31}$. [1]

Order 4×2, so $4 \times 2 = 8$ elements.

$a_{42} + a_{31} = -4 - 5 = -9$

Ans: 8 **elements;** $a_{42} + a_{31} = -9$

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2. Write the matrix $A = [a_{ij}]_{2\times 2}$ whose elements are $a_{ij} = \frac{(i+j)^2}{2}$. [2]

$a_{ij} = \frac{(i+j)^2}{2}$

$a_{11} = 2$, $a_{12} = \frac{9}{2}$

$a_{21} = \frac{9}{2}$, $a_{22} = 8$

$A = \begin{bmatrix} 2 & \frac{9}{2} \\ \frac{9}{2} & 8 \end{bmatrix}$

Ans: $A = \begin{bmatrix} 2 & \frac{9}{2} \\ \frac{9}{2} & 8 \end{bmatrix}$

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3. Name the type of the square matrix $\begin{bmatrix} -3 & 0 & 0 \\ -2 & 3 & 0 \\ -2 & -4 & -6 \end{bmatrix}$ (the most special type that fits). [1]

Here every element above the principal diagonal is 0 ($a_{ij} = 0$ for $i < j$).

So it is a lower triangular matrix.

Ans: A lower triangular matrix

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4. If $\operatorname{tr}(A) = 8$ and $\operatorname{tr}(B) = -8$, find $\operatorname{tr}(4A - 4B)$. [1]

$\operatorname{tr}(4A - 4B) = 4\operatorname{tr}(A) - 4\operatorname{tr}(B)$

$= 4(8) - 4(-8) = 64$

Ans: 64

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5. If $\begin{bmatrix} 3x - y & -2 \\ 6 & 3x + 2y \end{bmatrix} = \begin{bmatrix} 11 & -2 \\ 6 & 23 \end{bmatrix}$, find the values of x and y. [2]

Equate corresponding elements:

$3x - y = 11, \quad 3x + 2y = 23$

Solving, $x = 5$, $y = 4$.

Ans: $x = 5$, $y = 4$

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6. Find the matrix X if $3X + \begin{bmatrix} 2 & -6 & -5 \\ 0 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 8 & -12 & 10 \\ -3 & 16 & 8 \end{bmatrix}$. [2]

$3X = \begin{bmatrix} 8 & -12 & 10 \\ -3 & 16 & 8 \end{bmatrix} - \begin{bmatrix} 2 & -6 & -5 \\ 0 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 6 & -6 & 15 \\ -3 & 15 & 9 \end{bmatrix}$

$X = \frac{1}{3}\begin{bmatrix} 6 & -6 & 15 \\ -3 & 15 & 9 \end{bmatrix} = \begin{bmatrix} 2 & -2 & 5 \\ -1 & 5 & 3 \end{bmatrix}$

Ans: $X = \begin{bmatrix} 2 & -2 & 5 \\ -1 & 5 & 3 \end{bmatrix}$

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7. Find X and Y if $2X + Y = \begin{bmatrix} 11 & 6 \\ 9 & 18 \end{bmatrix}$ and $X + 2Y = \begin{bmatrix} 7 & 6 \\ 12 & 18 \end{bmatrix}$. [3]

Multiply the first by 2 and subtract the second: $3X = 2\begin{bmatrix} 11 & 6 \\ 9 & 18 \end{bmatrix} - \begin{bmatrix} 7 & 6 \\ 12 & 18 \end{bmatrix} = \begin{bmatrix} 15 & 6 \\ 6 & 18 \end{bmatrix}$, so $X = \begin{bmatrix} 5 & 2 \\ 2 & 6 \end{bmatrix}$.

Then $Y = \begin{bmatrix} 11 & 6 \\ 9 & 18 \end{bmatrix} - 2X = \begin{bmatrix} 1 & 2 \\ 5 & 6 \end{bmatrix}$.

Ans: $X = \begin{bmatrix} 5 & 2 \\ 2 & 6 \end{bmatrix}$, $Y = \begin{bmatrix} 1 & 2 \\ 5 & 6 \end{bmatrix}$

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8. If $A = \begin{bmatrix} -1 & 4 & 3 \\ -4 & -1 & -3 \end{bmatrix}$ and $B = \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix}$, find AB. [2]

A is 2×3 and B is 3×1, so AB is 2×1.

$c_{11} = (-1)(1) + (4)(-2) + (3)(2) = -3$

$c_{21} = (-4)(1) + (-1)(-2) + (-3)(2) = -8$

$AB = \begin{bmatrix} -3 \\ -8 \end{bmatrix}$

Ans: $AB = \begin{bmatrix} -3 \\ -8 \end{bmatrix}$

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9. If $A = \begin{bmatrix} -2 & 4 \\ -4 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & 5 \\ -4 & -2 \end{bmatrix}$, find AB and BA. Is $AB = BA$? [3]

$AB = \begin{bmatrix} -14 & -18 \\ -4 & -24 \end{bmatrix}$

$BA = \begin{bmatrix} -18 & 6 \\ 16 & -20 \end{bmatrix}$

The products differ, so $AB \neq BA$: matrix multiplication is not commutative.

Ans: $AB = \begin{bmatrix} -14 & -18 \\ -4 & -24 \end{bmatrix}$, $BA = \begin{bmatrix} -18 & 6 \\ 16 & -20 \end{bmatrix}$; $AB \neq BA$

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10. If $A = \begin{bmatrix} 3 & 1 \\ 1 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 4 & 4 \\ 1 & 0 \end{bmatrix}$, find $(A+B)^2$ and $A^2 + 2AB + B^2$. Are they equal? Why? [5]

$A + B = \begin{bmatrix} 7 & 5 \\ 2 & 1 \end{bmatrix}$, $(A+B)^2 = \begin{bmatrix} 59 & 40 \\ 16 & 11 \end{bmatrix}$

$A^2 = \begin{bmatrix} 10 & 4 \\ 4 & 2 \end{bmatrix}$, $B^2 = \begin{bmatrix} 20 & 16 \\ 4 & 4 \end{bmatrix}$

$AB = \begin{bmatrix} 13 & 12 \\ 5 & 4 \end{bmatrix}$, $BA = \begin{bmatrix} 16 & 8 \\ 3 & 1 \end{bmatrix}$

$A^2 + 2AB + B^2 = \begin{bmatrix} 56 & 44 \\ 18 & 14 \end{bmatrix}$

They differ because $(A+B)^2 = A^2 + AB + BA + B^2$ and here $AB \neq BA$.

Ans: $(A+B)^2 = \begin{bmatrix} 59 & 40 \\ 16 & 11 \end{bmatrix}$, $A^2 + 2AB + B^2 = \begin{bmatrix} 56 & 44 \\ 18 & 14 \end{bmatrix}$; **not equal since** $AB \neq BA$

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11. Find A^T for $A = \begin{bmatrix} 2 & 9 & -6 & -9 \end{bmatrix}$ and state its order. [1]

Rows of A become columns of A^T.

$$A^T = \begin{bmatrix} 2 \\ 9 \\ -6 \\ -9 \end{bmatrix}$$

Order 4×1.

Ans: $A^T = \begin{bmatrix} 2 \\ 9 \\ -6 \\ -9 \end{bmatrix}$, **order** 4×1

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12. If $A = \begin{bmatrix} -5 & -2 & -4 \\ -2 & -5 & -5 \end{bmatrix}$ and $B = \begin{bmatrix} -4 & -3 & -1 \\ 5 & -4 & -2 \end{bmatrix}$, verify that $(3A - B)^T = 3A^T - B^T$. [3]

$$3A - B = \begin{bmatrix} -11 & -3 & -11 \\ -11 & -11 & -13 \end{bmatrix}$$

$$(3A - B)^T = \begin{bmatrix} -11 & -11 \\ -3 & -11 \\ -11 & -13 \end{bmatrix}$$

$$3A^T - B^T = \begin{bmatrix} -15 & -6 \\ -6 & -15 \\ -12 & -15 \end{bmatrix} - \begin{bmatrix} -4 & 5 \\ -3 & -4 \\ -1 & -2 \end{bmatrix} = \begin{bmatrix} -11 & -11 \\ -3 & -11 \\ -11 & -13 \end{bmatrix}$$

The two are equal.

Ans: Both equal $\begin{bmatrix} -11 & -11 \\ -3 & -11 \\ -11 & -13 \end{bmatrix}$

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13. Find x if $\begin{bmatrix} 7 & 2 & -3x + 9 \\ 2 & 2 & -9 \\ -6 & -9 & -8 \end{bmatrix}$ is symmetric. [1]

Symmetric means $a_{ij} = a_{ji}$.

$a_{13} = a_{31} \Rightarrow -3x + 9 = -6$

$x = 5$

Ans: $x = 5$

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14. For $A = \begin{bmatrix} 7 & -1 & 5 \\ 5 & 3 & 4 \\ 6 & -3 & 5 \end{bmatrix}$, find $A - A^T$ and show that it is skew-symmetric. [2]

$$A^T = \begin{bmatrix} 7 & 5 & 6 \\ -1 & 3 & -3 \\ 5 & 4 & 5 \end{bmatrix}$$

$$A - A^T = \begin{bmatrix} 0 & -6 & -1 \\ 6 & 0 & 7 \\ 1 & -7 & 0 \end{bmatrix}$$

Its diagonal is zero and $a_{ij} = -a_{ji}$, so it is skew-symmetric.

Ans: $A - A^T = \begin{bmatrix} 0 & -6 & -1 \\ 6 & 0 & 7 \\ 1 & -7 & 0 \end{bmatrix}$

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15. Write $\begin{bmatrix} 7 & 6 & 3 \\ -3 & -3 & 1 \\ 3 & -6 & -4 \end{bmatrix}$ as $P+Q$ where P is symmetric and Q is skew-symmetric. [5]

$A^T = \begin{bmatrix} 7 & -3 & 3 \\ 6 & -3 & -6 \\ 3 & 1 & -4 \end{bmatrix}$

$A + A^T = \begin{bmatrix} 14 & 3 & 6 \\ 3 & -6 & -5 \\ 6 & -5 & -8 \end{bmatrix}$, $\quad P = \frac{1}{2}(A+A^T) = \begin{bmatrix} 7 & \frac{3}{2} & 3 \\ \frac{3}{2} & -3 & -\frac{5}{2} \\ 3 & -\frac{5}{2} & -4 \end{bmatrix}$

$A - A^T = \begin{bmatrix} 0 & 9 & 0 \\ -9 & 0 & 7 \\ 0 & -7 & 0 \end{bmatrix}$, $\quad Q = \frac{1}{2}(A-A^T) = \begin{bmatrix} 0 & \frac{9}{2} & 0 \\ -\frac{9}{2} & 0 & \frac{7}{2} \\ 0 & -\frac{7}{2} & 0 \end{bmatrix}$

$P^T = P$ and $Q^T = -Q$, and $P + Q = A$.

$A = \begin{bmatrix} 7 & \frac{3}{2} & 3 \\ \frac{3}{2} & -3 & -\frac{5}{2} \\ 3 & -\frac{5}{2} & -4 \end{bmatrix} + \begin{bmatrix} 0 & \frac{9}{2} & 0 \\ -\frac{9}{2} & 0 & \frac{7}{2} \\ 0 & -\frac{7}{2} & 0 \end{bmatrix}$

Ans: $\begin{bmatrix} 7 & \frac{3}{2} & 3 \\ \frac{3}{2} & -3 & -\frac{5}{2} \\ 3 & -\frac{5}{2} & -4 \end{bmatrix} + \begin{bmatrix} 0 & \frac{9}{2} & 0 \\ -\frac{9}{2} & 0 & \frac{7}{2} \\ 0 & -\frac{7}{2} & 0 \end{bmatrix}$

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16. Find the positive number k for which $k\begin{bmatrix} 2 & -6 & -3 \\ -3 & 2 & -6 \\ -6 & -3 & 2 \end{bmatrix}$ is orthogonal, and check one row and two columns of your answer. [5]

Let $A = kM$. Then $AA^T = k^2 MM^T$.

$MM^T = \begin{bmatrix} 49 & 0 & 0 \\ 0 & 49 & 0 \\ 0 & 0 & 49 \end{bmatrix} = 49I$

$AA^T = I \Rightarrow 49k^2 = 1 \Rightarrow k = \frac{1}{7}$

Row 1: $\frac{1}{49}(2^2 + (-6)^2 + (-3)^2) = 1$.

Columns 1, 2: $\frac{1}{49}((2)(-6) + (-3)(2) + (-6)(-3)) = 0$.

Ans: $k = \frac{1}{7}$

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17. Show that $A = \begin{bmatrix} 4 & -5 \\ 3 & -4 \end{bmatrix}$ is involutory. [2]

$A^2 = \begin{bmatrix} 4 & -5 \\ 3 & -4 \end{bmatrix}\begin{bmatrix} 4 & -5 \\ 3 & -4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$A^2 = I$, so A is involutory.

Ans: $A^2 = I$

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18. Find the transposed conjugate A^θ of $A = \begin{bmatrix} 7i-2 & -4-3i \\ -2-2i & 7+3i \end{bmatrix}$. [1]

$\bar{A} = \begin{bmatrix} -2-7i & 3i-4 \\ 2i-2 & 7-3i \end{bmatrix}$

$A^\theta = (\bar{A})^T = \begin{bmatrix} -2-7i & 2i-2 \\ 3i-4 & 7-3i \end{bmatrix}$

Ans: $A^\theta = \begin{bmatrix} -2-7i & 2i-2 \\ 3i-4 & 7-3i \end{bmatrix}$

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19. Show that $A = \begin{bmatrix} -1 & 6 \\ 3 & -2 \end{bmatrix}$ satisfies $A^2 - (a+d)A + (ad-bc)I = O$, writing the equation with numbers. [3]

$a + d = -3, \quad ad - bc = -16$

$A^2 = \begin{bmatrix} 19 & -18 \\ -9 & 22 \end{bmatrix}$

$A^2 - (-3)A + (-16)I = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

$A^2 + 3A - 16I = O$

Ans: $A^2 + 3A - 16I = O$

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20. Write the equation $A^3 - (\operatorname{tr} A)A^2 + (\ldots)A - |A|I = O$ satisfied by $A = \begin{bmatrix} -3 & -2 & -3 \\ 4 & 4 & 0 \\ 0 & 1 & 0 \end{bmatrix}$, finding every coefficient. [5]

$\operatorname{tr} A = -3 + 4 + 0 = 1$

$\begin{vmatrix} 4 & 0 \\ 1 & 0 \end{vmatrix} = 0, \quad \begin{vmatrix} -3 & -3 \\ 0 & 0 \end{vmatrix} = 0, \quad \begin{vmatrix} -3 & -2 \\ 4 & 4 \end{vmatrix} = -4$

Sum of the principal minors $= -4$.

$|A| = -12$

So $A^3 - A^2 - 4A + 12I = O$.

Ans: $A^3 - A^2 - 4A + 12I = O$
