EXPRESSWALA ACADEMY

Maths 1A: Algebra of Matrices

Question Paper, Set 10

Time: 1 hour 30 minutes | Maximum marks: 50

Name: _____ Class: _____ Date: _____

1. Answer all 20 questions. The marks for each question are shown in brackets.
2. Show your working for the 2, 3 and 5 mark questions.
3. The key sheet and worked solutions are at the end of this paper.

1. For $A = \begin{bmatrix} 4 & 5 & 3 \\ 8 & -1 & -1 \\ 9 & -1 & 7 \\ -5 & 9 & -8 \end{bmatrix}$, find the number of elements of A and $a_{13} + a_{21}$. [1 mark]

2. Write the matrix $A = [a_{ij}]_{2\times 2}$ whose elements are $a_{ij} = 4^i - j$. [2 marks]

3. Name the type of the square matrix $\begin{bmatrix} -6 & 0 & 0 \\ -6 & 1 & 0 \\ -1 & 5 & -5 \end{bmatrix}$ (the most special type that fits). [1 mark]

4. Find the trace of $\begin{bmatrix} -3 & 1 & 1 \\ 5 & 4 & -3 \\ 3 & 9 & 3 \end{bmatrix}$. [1 mark]

5. If $\begin{bmatrix} 3x - y & -6 \\ -8 & x + 3y \end{bmatrix} = \begin{bmatrix} -3 & -6 \\ -8 & -21 \end{bmatrix}$, find the values of x and y. [2 marks]

6. Find the matrix X if $3X + \begin{bmatrix} 6 & 7 & 2 \\ 1 & -1 & 7 \end{bmatrix} = \begin{bmatrix} 9 & 19 & 14 \\ 4 & -10 & 22 \end{bmatrix}$. [2 marks]

7. Find X and Y if $2X + Y = \begin{bmatrix} -3 & 3 \\ 4 & -7 \end{bmatrix}$ and $X + 2Y = \begin{bmatrix} -3 & -3 \\ 8 & -5 \end{bmatrix}$. [3 marks]

8. If $A = \begin{bmatrix} 3 & 4 & 0 \\ 4 & 3 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 4 & 3 \\ 0 & 2 \\ 5 & 5 \end{bmatrix}$, find AB. [2 marks]

9. If $A = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} -2 & -5 \\ -2 & 4 \end{bmatrix}$, find AB and BA. Is $AB = BA$? [3 marks]

10. If $A = \begin{bmatrix} 4 & 1 \\ 4 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} -3 & -3 \\ 4 & 0 \end{bmatrix}$, find $(A + B)^2$ and $A^2 + 2AB + B^2$. Are they equal? Why? [5 marks]

11. Find A^T for $A = \begin{bmatrix} 0 & -6 & 0 & -3 \\ 3 & 2 & -7 & 3 \end{bmatrix}$ and state its order. [1 mark]

12. If $A = \begin{bmatrix} 2 & -3 & -5 \\ -2 & 2 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 4 & -2 & -2 \\ 4 & 6 & 3 \end{bmatrix}$, verify that $(2A - B)^T = 2A^T - B^T$. [3 marks]

13. Find x if $\begin{bmatrix} 7 & 10 & x-9 \\ 10 & -1 & -8 \\ 4 & -8 & -1 \end{bmatrix}$ is symmetric. [1 mark]

14. For $A = \begin{bmatrix} -6 & 3 & -2 \\ 3 & -4 & -4 \\ 0 & 5 & -6 \end{bmatrix}$, find $A - A^T$ and show that it is skew-symmetric. [2 marks]

15. Write $\begin{bmatrix} 1 & -5 & 2 \\ 7 & 2 & 2 \\ 2 & -5 & -2 \end{bmatrix}$ as $P + Q$ where P is symmetric and Q is skew-symmetric. [5 marks]

16. Find the positive number k for which $k \begin{bmatrix} -3 & 6 & -2 \\ 2 & 3 & 6 \\ -6 & -2 & 3 \end{bmatrix}$ is orthogonal, and check one row and two columns of your answer. [5 marks]

17. Show that $A = \begin{bmatrix} -2 & -2 \\ 3 & 3 \end{bmatrix}$ is idempotent. [2 marks]

18. Find the transposed conjugate A^θ of $A = \begin{bmatrix} i-6 & 5+2i \\ 1-2i & 4-2i \end{bmatrix}$. [1 mark]

19. Show that $A = \begin{bmatrix} 4 & -2 \\ -5 & 4 \end{bmatrix}$ satisfies $A^2 - (a+d)A + (ad-bc)I = O$, writing the equation with numbers. [3 marks]

20. Write the equation $A^3 - (\operatorname{tr} A)A^2 + (\ldots)A - |A|I = O$ satisfied by $A = \begin{bmatrix} 2 & 2 & -1 \\ -3 & 2 & 0 \\ 2 & -3 & -1 \end{bmatrix}$, finding every coefficient. [5 marks]

Key sheet

Q	Answer
1	12 elements; $a_{13} + a_{21} = 11$
2	$A = \begin{bmatrix} 3 & 2 \\ 15 & 14 \end{bmatrix}$
3	A lower triangular matrix
4	4
5	$x = -3$, $y = -6$
6	$X = \begin{bmatrix} 1 & 4 & 4 \\ 1 & -3 & 5 \end{bmatrix}$
7	$X = \begin{bmatrix} -1 & 3 \\ 0 & -3 \end{bmatrix}$, $Y = \begin{bmatrix} -1 & -3 \\ 4 & -1 \end{bmatrix}$
8	$AB = \begin{bmatrix} 12 & 17 \\ 36 & 38 \end{bmatrix}$
9	$AB = \begin{bmatrix} -10 & 2 \\ -6 & 3 \end{bmatrix}$, $BA = \begin{bmatrix} -9 & -16 \\ 0 & 2 \end{bmatrix}$; $AB \neq BA$
10	$(A+B)^2 = \begin{bmatrix} -15 & -4 \\ 16 & -15 \end{bmatrix}$, $A^2 + 2AB + B^2 = \begin{bmatrix} 1 & -10 \\ -8 & -31 \end{bmatrix}$; not equal since $AB \neq BA$
11	$A^T = \begin{bmatrix} 0 & 3 \\ -6 & 2 \\ 0 & -7 \\ -3 & 3 \end{bmatrix}$, order 4×2
12	Both equal $\begin{bmatrix} 0 & -8 \\ -4 & -2 \\ -8 & 1 \end{bmatrix}$
13	$x = 13$
14	$A - A^T = \begin{bmatrix} 0 & 0 & -2 \\ 0 & 0 & -9 \\ 2 & 9 & 0 \end{bmatrix}$
15	$\begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & -\frac{3}{2} \\ 2 & -\frac{3}{2} & -2 \end{bmatrix} + \begin{bmatrix} 0 & -6 & 0 \\ 6 & 0 & \frac{7}{2} \\ 0 & -\frac{7}{2} & 0 \end{bmatrix}$
16	$k = \frac{1}{7}$
17	$A^2 = A$
18	$A^\theta = \begin{bmatrix} -6 - i & 1 + 2i \\ 5 - 2i & 4 + 2i \end{bmatrix}$
19	$A^2 - 8A + 6I = O$
20	$A^3 - 3A^2 + 8A + 15I = O$

Solutions

1. For $A = \begin{bmatrix} 4 & 5 & 3 \\ 8 & -1 & -1 \\ 9 & -1 & 7 \\ -5 & 9 & -8 \end{bmatrix}$, find the number of elements of A and $a_{13} + a_{21}$. [1]

Order 4×3, so $4 \times 3 = 12$ elements.

$a_{13} + a_{21} = 3 + 8 = 11$

Ans: 12 **elements;** $a_{13} + a_{21} = 11$

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2. Write the matrix $A = [a_{ij}]_{2\times 2}$ whose elements are $a_{ij} = 4^i - j$. [2]

$a_{ij} = 4^i - j$

$a_{11} = 3$, $a_{12} = 2$

$a_{21} = 15$, $a_{22} = 14$

$A = \begin{bmatrix} 3 & 2 \\ 15 & 14 \end{bmatrix}$

Ans: $A = \begin{bmatrix} 3 & 2 \\ 15 & 14 \end{bmatrix}$

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3. Name the type of the square matrix $\begin{bmatrix} -6 & 0 & 0 \\ -6 & 1 & 0 \\ -1 & 5 & -5 \end{bmatrix}$ (the most special type that fits). [1]

Here every element above the principal diagonal is 0 ($a_{ij} = 0$ for $i < j$).

So it is a lower triangular matrix.

Ans: A lower triangular matrix

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4. Find the trace of $\begin{bmatrix} -3 & 1 & 1 \\ 5 & 4 & -3 \\ 3 & 9 & 3 \end{bmatrix}$. [1]

$\text{tr}(A) = a_{11} + a_{22} + a_{33}$

$= -3 + 4 + 3 = 4$

Ans: 4

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5. If $\begin{bmatrix} 3x - y & -6 \\ -8 & x + 3y \end{bmatrix} = \begin{bmatrix} -3 & -6 \\ -8 & -21 \end{bmatrix}$, find the values of x and y. [2]

Equate corresponding elements:

$3x - y = -3, \quad x + 3y = -21$

Solving, $x = -3$, $y = -6$.

Ans: $x = -3$, $y = -6$

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6. Find the matrix X if $3X + \begin{bmatrix} 6 & 7 & 2 \\ 1 & -1 & 7 \end{bmatrix} = \begin{bmatrix} 9 & 19 & 14 \\ 4 & -10 & 22 \end{bmatrix}$. [2]

$3X = \begin{bmatrix} 9 & 19 & 14 \\ 4 & -10 & 22 \end{bmatrix} - \begin{bmatrix} 6 & 7 & 2 \\ 1 & -1 & 7 \end{bmatrix} = \begin{bmatrix} 3 & 12 & 12 \\ 3 & -9 & 15 \end{bmatrix}$

$X = \frac{1}{3}\begin{bmatrix} 3 & 12 & 12 \\ 3 & -9 & 15 \end{bmatrix} = \begin{bmatrix} 1 & 4 & 4 \\ 1 & -3 & 5 \end{bmatrix}$

Ans: $X = \begin{bmatrix} 1 & 4 & 4 \\ 1 & -3 & 5 \end{bmatrix}$

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7. Find X and Y if $2X + Y = \begin{bmatrix} -3 & 3 \\ 4 & -7 \end{bmatrix}$ and $X + 2Y = \begin{bmatrix} -3 & -3 \\ 8 & -5 \end{bmatrix}$. [3]

Multiply the first by 2 and subtract the second: $3X = 2\begin{bmatrix} -3 & 3 \\ 4 & -7 \end{bmatrix} - \begin{bmatrix} -3 & -3 \\ 8 & -5 \end{bmatrix} = \begin{bmatrix} -3 & 9 \\ 0 & -9 \end{bmatrix}$, so $X = \begin{bmatrix} -1 & 3 \\ 0 & -3 \end{bmatrix}$.

Then $Y = \begin{bmatrix} -3 & 3 \\ 4 & -7 \end{bmatrix} - 2X = \begin{bmatrix} -1 & -3 \\ 4 & -1 \end{bmatrix}$.

Ans: $X = \begin{bmatrix} -1 & 3 \\ 0 & -3 \end{bmatrix}$, $Y = \begin{bmatrix} -1 & -3 \\ 4 & -1 \end{bmatrix}$

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8. If $A = \begin{bmatrix} 3 & 4 & 0 \\ 4 & 3 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 4 & 3 \\ 0 & 2 \\ 5 & 5 \end{bmatrix}$, find AB. [2]

A is 2×3 and B is 3×2, so AB is 2×2.

$c_{11} = (3)(4) + (4)(0) + (0)(5) = 12$

$c_{12} = (3)(3) + (4)(2) + (0)(5) = 17$

$c_{21} = (4)(4) + (3)(0) + (4)(5) = 36$

$c_{22} = (4)(3) + (3)(2) + (4)(5) = 38$

$AB = \begin{bmatrix} 12 & 17 \\ 36 & 38 \end{bmatrix}$

Ans: $AB = \begin{bmatrix} 12 & 17 \\ 36 & 38 \end{bmatrix}$

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9. If $A = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} -2 & -5 \\ -2 & 4 \end{bmatrix}$, find AB and BA. Is $AB = BA$? [3]

$AB = \begin{bmatrix} -10 & 2 \\ -6 & 3 \end{bmatrix}$

$BA = \begin{bmatrix} -9 & -16 \\ 0 & 2 \end{bmatrix}$

The products differ, so $AB \neq BA$: matrix multiplication is not commutative.

Ans: $AB = \begin{bmatrix} -10 & 2 \\ -6 & 3 \end{bmatrix}$, $BA = \begin{bmatrix} -9 & -16 \\ 0 & 2 \end{bmatrix}$; $AB \neq BA$

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10. If $A = \begin{bmatrix} 4 & 1 \\ 4 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} -3 & -3 \\ 4 & 0 \end{bmatrix}$, find $(A+B)^2$ and $A^2 + 2AB + B^2$. Are they equal? Why? [5]

$A + B = \begin{bmatrix} 1 & -2 \\ 8 & 1 \end{bmatrix}$, $(A+B)^2 = \begin{bmatrix} -15 & -4 \\ 16 & -15 \end{bmatrix}$

$A^2 = \begin{bmatrix} 20 & 5 \\ 20 & 5 \end{bmatrix}$, $B^2 = \begin{bmatrix} -3 & 9 \\ -12 & -12 \end{bmatrix}$

$AB = \begin{bmatrix} -8 & -12 \\ -8 & -12 \end{bmatrix}$, $BA = \begin{bmatrix} -24 & -6 \\ 16 & 4 \end{bmatrix}$

$A^2 + 2AB + B^2 = \begin{bmatrix} 1 & -10 \\ -8 & -31 \end{bmatrix}$

They differ because $(A+B)^2 = A^2 + AB + BA + B^2$ and here $AB \neq BA$.

Ans: $(A+B)^2 = \begin{bmatrix} -15 & -4 \\ 16 & -15 \end{bmatrix}$, $A^2 + 2AB + B^2 = \begin{bmatrix} 1 & -10 \\ -8 & -31 \end{bmatrix}$; **not equal since** $AB \neq BA$

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11. Find A^T for $A = \begin{bmatrix} 0 & -6 & 0 & -3 \\ 3 & 2 & -7 & 3 \end{bmatrix}$ and state its order. [1]

Rows of A become columns of A^T.

$A^T = \begin{bmatrix} 0 & 3 \\ -6 & 2 \\ 0 & -7 \\ -3 & 3 \end{bmatrix}$

Order 4×2.

Ans: $A^T = \begin{bmatrix} 0 & 3 \\ -6 & 2 \\ 0 & -7 \\ -3 & 3 \end{bmatrix}$, **order** 4×2

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12. If $A = \begin{bmatrix} 2 & -3 & -5 \\ -2 & 2 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 4 & -2 & -2 \\ 4 & 6 & 3 \end{bmatrix}$, verify that $(2A - B)^T = 2A^T - B^T$. [3]

$2A - B = \begin{bmatrix} 0 & -4 & -8 \\ -8 & -2 & 1 \end{bmatrix}$

$(2A - B)^T = \begin{bmatrix} 0 & -8 \\ -4 & -2 \\ -8 & 1 \end{bmatrix}$

$2A^T - B^T = \begin{bmatrix} 4 & -4 \\ -6 & 4 \\ -10 & 4 \end{bmatrix} - \begin{bmatrix} 4 & 4 \\ -2 & 6 \\ -2 & 3 \end{bmatrix} = \begin{bmatrix} 0 & -8 \\ -4 & -2 \\ -8 & 1 \end{bmatrix}$

The two are equal.

Ans: Both equal $\begin{bmatrix} 0 & -8 \\ -4 & -2 \\ -8 & 1 \end{bmatrix}$

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13. Find x if $\begin{bmatrix} 7 & 10 & x-9 \\ 10 & -1 & -8 \\ 4 & -8 & -1 \end{bmatrix}$ is symmetric. [1]

Symmetric means $a_{ij} = a_{ji}$.

$a_{13} = a_{31} \Rightarrow x - 9 = 4$

$x = 13$

Ans: $x = 13$

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14. For $A = \begin{bmatrix} -6 & 3 & -2 \\ 3 & -4 & -4 \\ 0 & 5 & -6 \end{bmatrix}$, find $A - A^T$ and show that it is skew-symmetric. [2]

$A^T = \begin{bmatrix} -6 & 3 & 0 \\ 3 & -4 & 5 \\ -2 & -4 & -6 \end{bmatrix}$

$A - A^T = \begin{bmatrix} 0 & 0 & -2 \\ 0 & 0 & -9 \\ 2 & 9 & 0 \end{bmatrix}$

Its diagonal is zero and $a_{ij} = -a_{ji}$, so it is skew-symmetric.

Ans: $A - A^T = \begin{bmatrix} 0 & 0 & -2 \\ 0 & 0 & -9 \\ 2 & 9 & 0 \end{bmatrix}$

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15. Write $\begin{bmatrix} 1 & -5 & 2 \\ 7 & 2 & 2 \\ 2 & -5 & -2 \end{bmatrix}$ as $P + Q$ where P is symmetric and Q is skew-symmetric. [5]

$A^T = \begin{bmatrix} 1 & 7 & 2 \\ -5 & 2 & -5 \\ 2 & 2 & -2 \end{bmatrix}$

$A + A^T = \begin{bmatrix} 2 & 2 & 4 \\ 2 & 4 & -3 \\ 4 & -3 & -4 \end{bmatrix}$, $\quad P = \frac{1}{2}(A + A^T) = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & -\frac{3}{2} \\ 2 & -\frac{3}{2} & -2 \end{bmatrix}$

$A - A^T = \begin{bmatrix} 0 & -12 & 0 \\ 12 & 0 & 7 \\ 0 & -7 & 0 \end{bmatrix}$, $\quad Q = \frac{1}{2}(A - A^T) = \begin{bmatrix} 0 & -6 & 0 \\ 6 & 0 & \frac{7}{2} \\ 0 & -\frac{7}{2} & 0 \end{bmatrix}$

$P^T = P$ and $Q^T = -Q$, and $P + Q = A$.

$A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & -\frac{3}{2} \\ 2 & -\frac{3}{2} & -2 \end{bmatrix} + \begin{bmatrix} 0 & -6 & 0 \\ 6 & 0 & \frac{7}{2} \\ 0 & -\frac{7}{2} & 0 \end{bmatrix}$

Ans: $\begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & -\frac{3}{2} \\ 2 & -\frac{3}{2} & -2 \end{bmatrix} + \begin{bmatrix} 0 & -6 & 0 \\ 6 & 0 & \frac{7}{2} \\ 0 & -\frac{7}{2} & 0 \end{bmatrix}$

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16. Find the positive number k for which $k \begin{bmatrix} -3 & 6 & -2 \\ 2 & 3 & 6 \\ -6 & -2 & 3 \end{bmatrix}$ is orthogonal, and check one row and two columns of your answer. [5]

Let $A = kM$. Then $AA^T = k^2 MM^T$.

$MM^T = \begin{bmatrix} 49 & 0 & 0 \\ 0 & 49 & 0 \\ 0 & 0 & 49 \end{bmatrix} = 49I$

$AA^T = I \Rightarrow 49k^2 = 1 \Rightarrow k = \frac{1}{7}$

Row 1: $\frac{1}{49}((-3)^2 + 6^2 + (-2)^2) = 1$.

Columns 1, 2: $\frac{1}{49}((-3)(6) + (2)(3) + (-6)(-2)) = 0$.

Ans: $k = \frac{1}{7}$

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17. Show that $A = \begin{bmatrix} -2 & -2 \\ 3 & 3 \end{bmatrix}$ is idempotent. [2]

$A^2 = \begin{bmatrix} -2 & -2 \\ 3 & 3 \end{bmatrix}\begin{bmatrix} -2 & -2 \\ 3 & 3 \end{bmatrix} = \begin{bmatrix} -2 & -2 \\ 3 & 3 \end{bmatrix}$

$A^2 = A$, so A is idempotent.

Ans: $A^2 = A$

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18. Find the transposed conjugate A^θ of $A = \begin{bmatrix} i-6 & 5+2i \\ 1-2i & 4-2i \end{bmatrix}$. [1]

$\bar{A} = \begin{bmatrix} -6-i & 5-2i \\ 1+2i & 4+2i \end{bmatrix}$

$A^\theta = (\bar{A})^T = \begin{bmatrix} -6-i & 1+2i \\ 5-2i & 4+2i \end{bmatrix}$

Ans: $A^\theta = \begin{bmatrix} -6-i & 1+2i \\ 5-2i & 4+2i \end{bmatrix}$

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19. Show that $A = \begin{bmatrix} 4 & -2 \\ -5 & 4 \end{bmatrix}$ satisfies $A^2 - (a+d)A + (ad-bc)I = O$, writing the equation with numbers. [3]

$a + d = 8, \quad ad - bc = 6$

$A^2 = \begin{bmatrix} 26 & -16 \\ -40 & 26 \end{bmatrix}$

$A^2 - (8)A + (6)I = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

$A^2 - 8A + 6I = O$

Ans: $A^2 - 8A + 6I = O$

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20. Write the equation $A^3 - (\operatorname{tr} A)A^2 + (\ldots)A - |A|I = O$ satisfied by $A = \begin{bmatrix} 2 & 2 & -1 \\ -3 & 2 & 0 \\ 2 & -3 & -1 \end{bmatrix}$, finding every coefficient. [5]

$\operatorname{tr} A = 2 + 2 - 1 = 3$

$\begin{vmatrix} 2 & 0 \\ -3 & -1 \end{vmatrix} = -2, \quad \begin{vmatrix} 2 & -1 \\ 2 & -1 \end{vmatrix} = 0, \quad \begin{vmatrix} 2 & 2 \\ -3 & 2 \end{vmatrix} = 10$

Sum of the principal minors $= 8$.

$|A| = -15$

So $A^3 - 3A^2 + 8A + 15I = O$.

Ans: $A^3 - 3A^2 + 8A + 15I = O$