EXPRESSWALA ACADEMY

Maths 1A: Addition of Vectors

Question Paper, Set 09

Time: 1 hour 30 minutes | Maximum marks: 50

Name: _____ Class: _____ Date: _____

1. Answer all 20 questions. The marks for each question are shown in brackets.
2. Show your working for the 2, 3 and 5 mark questions.
3. The key sheet and worked solutions are at the end of this paper.

1. Find the unit vector along $3\bar{i} + 2\bar{j} - 4\bar{k}$. [1 mark]

2. Find the vectors of magnitude 42 that are parallel to $\bar{a} = 19\bar{i} - 8\bar{j} - 4\bar{k}$. [2 marks]

3. The position vectors of A and B are $-2\bar{i} - 4\bar{j} - 4\bar{k}$ and $4\bar{i} + \bar{j} - \bar{k}$. Find $\overline{AB}$. [1 mark]

4. If $(\bar{a}, \bar{b}) = 130^\circ$, find $(-3\bar{a}, -5\bar{b})$. [1 mark]

5. Find the magnitude of $\overline{OP}$, where $P = (-6, -1, 6)$. [1 mark]

6. Find the direction cosines of $\overline{AB}$, where $A = (-2, 5, 2)$ and $B = (-8, -3, 2)$. [2 marks]

7. Find p if the vectors $-4\bar{i} - 8\bar{j} + p\bar{k}$ and $-2\bar{i} - 4\bar{j} + 2\bar{k}$ are collinear. [2 marks]

8. Show that the points $A(5, 1, -1)$, $B(4, 4, -1)$ and $C(7, -5, -1)$ are collinear. [3 marks]

9. Write $\bar{r} = 6\bar{i} - 4\bar{j} - 8\bar{k}$ in terms of the non-coplanar vectors $\bar{a} = -2\bar{i} + 2\bar{j} + 2\bar{k}$, $\bar{b} = \bar{i} - \bar{k}$ and $\bar{c} = -\bar{i} + \bar{j}$. [5 marks]

10. Find p so that the vectors $3\bar{i} + 3\bar{k}$, $\bar{i} + 2\bar{j} - \bar{k}$ and $-5\bar{i} + 2\bar{j} + p\bar{k}$ are linearly dependent. [5 marks]

11. Find the midpoint of the segment joining $A(2, 5, 7)$ and $B(-6, 5, 2)$. [1 mark]

12. Find the point that divides the segment joining $A(4, -4, 0)$ and $B(-5, 5, 0)$ internally in the ratio $2 : 1$. [2 marks]

13. The point $C(-1, 1, -4)$ divides AB internally in the ratio $2 : 1$, and $A = (3, 5, -4)$. Find B. [3 marks]

14. Find the incentre of the triangle whose vertices are $A(3, 16, 3)$, $B(3, 1, 3)$ and $C(3, 1, -5)$. [5 marks]

15. Find the vector and cartesian equations of the line through the origin and the point $P(3, -1, -4)$. [3 marks]

16. Write the vector equation of the plane through the points $\bar{i} - 2\bar{k}$ and $-2\bar{i} + \bar{j} + 2\bar{k}$ parallel to $-2\bar{i} - \bar{j} + 4\bar{k}$, and find its point with $s = 1$, $t = 1$. [2 marks]

17. Find a vector along the internal bisector of the angle between $\bar{a} = -7\bar{i} + 6\bar{j} - 6\bar{k}$ and $\bar{b} = 6\bar{i} - 6\bar{j} - 7\bar{k}$. [2 marks]

18. Prove that the centroid of a tetrahedron divides the line joining a vertex to the centroid of the opposite face in the ratio $3:1$. [5 marks]

19. Find the components of $\bar{a} = 5\bar{i} + 3\bar{j} + 5\bar{k}$ when the coordinate axes are rotated about the z-axis through 60°. [3 marks]

20. In triangle ABC, the lines joining a point P to A, B, C meet BC, CA, AB at D, E, F. If $\frac{BD}{DC} = \frac{1}{3}$ and $\frac{CE}{EA} = \frac{5}{2}$, find $\frac{AF}{FB}$. [1 mark]

Key sheet

Q	Answer
1	$\frac{1}{\sqrt{29}}(3\bar{i}+2\bar{j}-4\bar{k})$
2	$\pm(38\bar{i}-16\bar{j}-8\bar{k})$
3	$6\bar{i}+5\bar{j}+3\bar{k}$
4	130°
5	$\sqrt{73}$
6	$(-\frac{3}{5},-\frac{4}{5},0)$
7	$p=4$
8	$\overline{AC}=-2\,\overline{AB}$, so A, B, C are collinear
9	$\bar{r}=-3\bar{a}+2\bar{b}+2\bar{c}$ ($x=-3$, $y=2$, $z=2$)
10	$p=-7$
11	$(-2,5,\frac{9}{2})$
12	$(-2,2,0)$
13	$B=(-3,-1,-4)$
14	$(3,4,0)$
15	$\bar{r}=t(3\bar{i}-\bar{j}-4\bar{k})$; $\frac{x}{3}=\frac{y}{-1}=\frac{z}{-4}$
16	$\bar{r}=(1-s)(\bar{i}-2\bar{k})+s(-2\bar{i}+\bar{j}+2\bar{k})+t(-2\bar{i}-\bar{j}+4\bar{k})$; point $(-4,0,6)$
17	$-\frac{1}{11}\bar{i}-\frac{13}{11}\bar{k}$ (or any multiple)
18	Proved (see the working)
19	$(\frac{5}{2}+\frac{3\sqrt{3}}{2},\frac{3}{2}-\frac{5\sqrt{3}}{2},5)$
20	$\frac{6}{5}$

Solutions

1. Find the unit vector along $3\bar{i} + 2\bar{j} - 4\bar{k}$. [1]

$|3\bar{i} + 2\bar{j} - 4\bar{k}| = \sqrt{9 + 4 + 16} = \sqrt{29}$

$\text{Unit vector} = \frac{1}{\sqrt{29}}(3\bar{i} + 2\bar{j} - 4\bar{k})$

Ans: $\frac{1}{\sqrt{29}}(3\bar{i} + 2\bar{j} - 4\bar{k})$

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2. Find the vectors of magnitude 42 that are parallel to $\bar{a} = 19\bar{i} - 8\bar{j} - 4\bar{k}$. [2]

$|\bar{a}| = \sqrt{361 + 64 + 16} = 21$

$\text{Required vectors} = \pm\frac{42\,\bar{a}}{|\bar{a}|} = \pm\frac{42}{21}(19\bar{i} - 8\bar{j} - 4\bar{k})$

$= \pm(38\bar{i} - 16\bar{j} - 8\bar{k})$

Ans: $\pm(38\bar{i} - 16\bar{j} - 8\bar{k})$

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3. The position vectors of A and B are $-2\bar{i} - 4\bar{j} - 4\bar{k}$ and $4\bar{i} + \bar{j} - \bar{k}$. Find $\overline{AB}$. [1]

$\overline{AB} = \overline{OB} - \overline{OA}$

$= (4\bar{i} + \bar{j} - \bar{k}) - (-2\bar{i} - 4\bar{j} - 4\bar{k}) = 6\bar{i} + 5\bar{j} + 3\bar{k}$

Ans: $6\bar{i} + 5\bar{j} + 3\bar{k}$

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4. If $(\bar{a}, \bar{b}) = 130^\circ$, find $(-3\bar{a}, -5\bar{b})$. [1]

The scalars -3 and -5 have the same sign.

$(-3\bar{a}, -5\bar{b}) = (\bar{a}, \bar{b}) = 130^\circ$

Ans: 130°

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5. Find the magnitude of $\overline{OP}$, where $P = (-6, -1, 6)$. [1]

$\overline{OP} = -6\bar{i} - \bar{j} + 6\bar{k}$

$|\overline{OP}| = \sqrt{36 + 1 + 36} = \sqrt{73}$

Ans: $\sqrt{73}$

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6. Find the direction cosines of $\overline{AB}$, where $A = (-2, 5, 2)$ and $B = (-8, -3, 2)$. [2]

$\overline{AB} = -6\bar{i} - 8\bar{j}$

$|\overline{AB}| = \sqrt{36 + 64 + 0} = 10$

$\text{Direction cosines} = (-\frac{3}{5}, -\frac{4}{5}, 0)$

Ans: $(-\frac{3}{5}, -\frac{4}{5}, 0)$

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7. Find p if the vectors $-4\bar{i} - 8\bar{j} + p\bar{k}$ and $-2\bar{i} - 4\bar{j} + 2\bar{k}$ are collinear. [2]

Collinear vectors have proportional components.

$\frac{-4}{-2} = \frac{p}{2}$

$\Rightarrow p = 4$

Ans: $p = 4$

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8. Show that the points $A(5, 1, -1)$, $B(4, 4, -1)$ and $C(7, -5, -1)$ are collinear. [3]

$\overline{AB} = \overline{OB} - \overline{OA} = -\bar{i} + 3\bar{j}$

$\overline{AC} = \overline{OC} - \overline{OA} = 2\bar{i} - 6\bar{j}$

$\overline{AC} = -2\,\overline{AB}$

$\overline{AB}$ and $\overline{AC}$ are parallel and have the common point A, so A, B, C are collinear.

Ans: $\overline{AC} = -2\,\overline{AB}$**, so** A, B, C **are collinear**

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9. Write $\bar{r} = 6\bar{i} - 4\bar{j} - 8\bar{k}$ in terms of the non-coplanar vectors $\bar{a} = -2\bar{i} + 2\bar{j} + 2\bar{k}$, $\bar{b} = \bar{i} - \bar{k}$ and $\bar{c} = -\bar{i} + \bar{j}$. [5]

Let $6\bar{i} - 4\bar{j} - 8\bar{k} = x(-2\bar{i} + 2\bar{j} + 2\bar{k}) + y(\bar{i} - \bar{k}) + z(-\bar{i} + \bar{j})$.

Comparing the coefficients of $\bar{i}, \bar{j}, \bar{k}$:

$-2x + y - z = 6$

$2x + z = -4$

$2x - y = -8$

Solving by Cramer's rule:

$$\Delta = \begin{vmatrix} -2 & 1 & -1 \\ 2 & 0 & 1 \\ 2 & -1 & 0 \end{vmatrix} = 2$$

$$\Delta_1 = \begin{vmatrix} 6 & 1 & -1 \\ -4 & 0 & 1 \\ -8 & -1 & 0 \end{vmatrix} = -6, \quad x = \frac{\Delta_1}{\Delta} = -3$$

$$\Delta_2 = \begin{vmatrix} -2 & 6 & -1 \\ 2 & -4 & 1 \\ 2 & -8 & 0 \end{vmatrix} = 4, \quad y = \frac{\Delta_2}{\Delta} = 2$$

$$\Delta_3 = \begin{vmatrix} -2 & 1 & 6 \\ 2 & 0 & -4 \\ 2 & -1 & -8 \end{vmatrix} = 4, \quad z = \frac{\Delta_3}{\Delta} = 2$$

$\therefore \bar{r} = -3\bar{a} + 2\bar{b} + 2\bar{c}$

Ans: $\bar{r} = -3\bar{a} + 2\bar{b} + 2\bar{c}$ **(**$x = -3$**,** $y = 2$**,** $z = 2$**)**

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10. Find p so that the vectors $3\bar{i} + 3\bar{k}$, $\bar{i} + 2\bar{j} - \bar{k}$ and $-5\bar{i} + 2\bar{j} + p\bar{k}$ are linearly dependent. [5]

Three vectors are linearly dependent (coplanar) when the determinant of their components is zero.

$$\begin{vmatrix} 3 & 0 & 3 \\ 1 & 2 & -1 \\ -5 & 2 & p \end{vmatrix} = 0$$

$\Rightarrow 6p + 42 = 0$

$\Rightarrow p = -7$

Ans: $p = -7$

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11. Find the midpoint of the segment joining $A(2, 5, 7)$ and $B(-6, 5, 2)$. [1]

$\text{Midpoint} = \frac{\bar{a} + \bar{b}}{2} = \frac{(-4, 10, 9)}{2}$

$= (-2, 5, \frac{9}{2})$

Ans: $(-2, 5, \frac{9}{2})$

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12. Find the point that divides the segment joining $A(4,-4,0)$ and $B(-5,5,0)$ internally in the ratio $2:1$. [2]

$\bar{r} = \frac{m\bar{b}+n\bar{a}}{m+n}$
$= \frac{2(-5,5,0)+1(4,-4,0)}{2+1}$
$= \frac{(-6,6,0)}{3} = (-2,2,0)$

Ans: $(-2,2,0)$

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13. The point $C(-1,1,-4)$ divides AB internally in the ratio $2:1$, and $A=(3,5,-4)$. Find B. [3]

$\bar{c} = \frac{2\bar{b}+1\bar{a}}{3} \Rightarrow \bar{b} = \frac{3\bar{c}-1\bar{a}}{2}$
$= \frac{(-3,3,-12)-(3,5,-4)}{2}$
$= (-3,-1,-4)$

Ans: $B=(-3,-1,-4)$

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14. Find the incentre of the triangle whose vertices are $A(3,16,3)$, $B(3,1,3)$ and $C(3,1,-5)$. [5]

$a = BC = \sqrt{0+0+64} = 8$
$b = CA = \sqrt{0+225+64} = 17$
$c = AB = \sqrt{0+225+0} = 15$
$\text{Incentre } I = \frac{a\bar{a}+b\bar{b}+c\bar{c}}{a+b+c}$
$= \frac{8(3,16,3)+17(3,1,3)+15(3,1,-5)}{40}$
$= \frac{(120,160,0)}{40} = (3,4,0)$

Ans: $(3,4,0)$

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15. Find the vector and cartesian equations of the line through the origin and the point $P(3,-1,-4)$. [3]

$\bar{r} = t\,\overrightarrow{OP} = t(3\bar{i}-\bar{j}-4\bar{k})$
$\text{Cartesian: } \frac{x}{3} = \frac{y}{-1} = \frac{z}{-4}$

Ans: $\bar{r} = t(3\bar{i}-\bar{j}-4\bar{k})$; $\frac{x}{3} = \frac{y}{-1} = \frac{z}{-4}$

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16. Write the vector equation of the plane through the points $\bar{i}-2\bar{k}$ and $-2\bar{i}+\bar{j}+2\bar{k}$ parallel to $-2\bar{i}-\bar{j}+4\bar{k}$, and find its point with $s=1$, $t=1$. [2]

$\bar{r} = (1-s)\bar{a} + s\bar{b} + t\bar{c}$
$\bar{r} = (1-s)(\bar{i}-2\bar{k}) + s(-2\bar{i}+\bar{j}+2\bar{k}) + t(-2\bar{i}-\bar{j}+4\bar{k})$
$s=1,\ t=1:\ \bar{r} = (-4,0,6)$

Ans: $\bar{r} = (1-s)(\bar{i}-2\bar{k}) + s(-2\bar{i}+\bar{j}+2\bar{k}) + t(-2\bar{i}-\bar{j}+4\bar{k})$; **point** $(-4,0,6)$

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17. Find a vector along the internal bisector of the angle between $\bar{a} = -7\bar{i}+6\bar{j}-6\bar{k}$ and $\bar{b} = 6\bar{i}-6\bar{j}-7\bar{k}$. [2]

$|\bar{a}| = \sqrt{49+36+36} = 11, \quad |\bar{b}| = \sqrt{36+36+49} = 11$
$\frac{\bar{a}}{|\bar{a}|} + \frac{\bar{b}}{|\bar{b}|} = \frac{1}{11}(-7\bar{i}+6\bar{j}-6\bar{k}) + \frac{1}{11}(6\bar{i}-6\bar{j}-7\bar{k})$
$= -\frac{1}{11}\bar{i} - \frac{13}{11}\bar{k}$

Any non-zero multiple of this vector is also along the internal bisector.

Ans: $-\frac{1}{11}\bar{i} - \frac{13}{11}\bar{k}$ **(or any multiple)**

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18. Prove that the centroid of a tetrahedron divides the line joining a vertex to the centroid of the opposite face in the ratio $3:1$. [5]

Let the vertices A, B, C, D have position vectors $\bar{a}, \bar{b}, \bar{c}, \bar{d}$.

Centroid of face BCD : $\bar{g}_1 = \frac{\bar{b}+\bar{c}+\bar{d}}{3}$

Point dividing AG_1 in $3:1 = \frac{3\bar{g}_1+1\cdot\bar{a}}{4} = \frac{\bar{a}+\bar{b}+\bar{c}+\bar{d}}{4}$

This is the centroid of the tetrahedron.

The expression is symmetric in $\bar{a}, \bar{b}, \bar{c}, \bar{d}$, so the same holds for every vertex.

Ans: Proved (see the working)

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19. Find the components of $\bar{a} = 5\bar{i} + 3\bar{j} + 5\bar{k}$ when the coordinate axes are rotated about the z-axis through $60°$. [3]

About the z-axis through α the new components are $(a_1\cos\alpha + a_2\sin\alpha,\ -a_1\sin\alpha + a_2\cos\alpha,\ a_3)$.

$\cos 60° = \frac{1}{2}, \quad \sin 60° = \frac{\sqrt{3}}{2}$

Substituting $a_1 = 5$, $a_2 = 3$, $a_3 = 5$:

$(5\cdot\frac{1}{2} + 3\cdot\frac{\sqrt{3}}{2},\ -5\cdot\frac{\sqrt{3}}{2} + 3\cdot\frac{1}{2},\ 5)$

$= (\frac{5}{2} + \frac{3\sqrt{3}}{2}, \frac{3}{2} - \frac{5\sqrt{3}}{2}, 5)$

Ans: $(\frac{5}{2} + \frac{3\sqrt{3}}{2}, \frac{3}{2} - \frac{5\sqrt{3}}{2}, 5)$

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20. In triangle ABC, the lines joining a point P to A, B, C meet BC, CA, AB at D, E, F. If $\frac{BD}{DC} = \frac{1}{3}$ and $\frac{CE}{EA} = \frac{5}{2}$, find $\frac{AF}{FB}$. [1]

By Ceva's theorem $\frac{BD}{DC}\cdot\frac{CE}{EA}\cdot\frac{AF}{FB} = 1$

$\frac{AF}{FB} = \frac{1}{\frac{1}{3}\times\frac{5}{2}} = \frac{6}{5}$

Ans: $\frac{6}{5}$
