EXPRESSWALA ACADEMY

# Maths 1A: Addition of Vectors

## Question Paper, Set 04

Time: 1 hour 30 minutes | Maximum marks: 50

Name: _____ Class: _____ Date: _____

1. Answer all 20 questions. The marks for each question are shown in brackets.
2. Show your working for the 2, 3 and 5 mark questions.
3. The key sheet and worked solutions are at the end of this paper.

**1.** Find the unit vector in the direction of $\bar{a} = 16\bar{i} + 12\bar{j} - 15\bar{k}$. [1 mark]

**2.** If $\bar{a} = -\bar{i} - 3\bar{j} - 3\bar{k}$ and $\bar{b} = -\bar{i} + 13\bar{j} + 14\bar{k}$, find the vector of magnitude $30$ in the direction of $\bar{a} + \bar{b}$. [2 marks]

**3.** If $\overline{OC} = -3\bar{i} + 2\bar{j} + 4\bar{k}$ and $\overline{OD} = -2\bar{i} - 3\bar{j} - 3\bar{k}$, find $\overline{CD}$. [1 mark]

**4.** If $(\bar{a}, \bar{b}) = 75^\circ$, find $(5\bar{a}, -2\bar{b})$. [1 mark]

**5.** Find the distance $|\overline{AB}|$ between $A(-1, -1, 0)$ and $B(0, 5, 3)$. [1 mark]

**6.** Find the direction cosines of the vector $4\bar{i} - 8\bar{j} - \bar{k}$. [2 marks]

**7.** Find $p$ if the vectors $18\bar{i} + p\bar{j} - 12\bar{k}$ and $6\bar{i} + 5\bar{j} - 4\bar{k}$ are collinear. [2 marks]

**8.** Find $p$ so that the points $A(-4, -1, -1)$, $B(-6, 2, 0)$ and $C(p, 11, 3)$ are collinear. [3 marks]

**9.** Express $\bar{r} = 3\bar{i} + 5\bar{j} - 7\bar{k}$ as a linear combination of $\bar{a} = 3\bar{i} - \bar{j} - \bar{k}$, $\bar{b} = 2\bar{i} - \bar{k}$ and $\bar{c} = -\bar{i} - \bar{j} + 2\bar{k}$. [5 marks]

**10.** Show that the points $A(3, -2, 1)$, $B(5, -3, 1)$, $C(5, -5, 4)$ and $D(5, -1, -2)$ are coplanar. [5 marks]

**11.** The midpoint of $AB$ is $(-\frac{1}{2}, 5, \frac{7}{2})$ and $A = (2, 5, 5)$. Find $B$. [1 mark]

**12.** Find the point that divides the segment joining $A(5, -3, -1)$ and $B(6, -1, -5)$ externally in the ratio $2 : 3$. [2 marks]

**13.** Find the points of trisection of the segment joining $A(-4, 5, 1)$ and $B(-4, -1, 4)$. [3 marks]

**14.** Find the incentre of the triangle whose vertices are $A(22, 1, 3)$, $B(13, 1, 15)$ and $C(4, 1, 3)$. [5 marks]

**15.** Find the vector and cartesian equations of the line through $A(2, 2, 2)$ and $B(1, -1, 5)$. [3 marks]

**16.** Write the vector equation of the plane through the points $A(1, 0, -2)$, $B(3, 4, 3)$ and $C(-1, 3, 2)$, and find its point with $s = 1$, $t = 2$. [2 marks]

**17.** Find a vector along the external bisector of the angle between $\bar{a} = -2\bar{i} - 6\bar{j} - 3\bar{k}$ and $\bar{b} = -4\bar{i} + 3\bar{j}$. [2 marks]

**18.** $ABCD$ is a parallelogram and $F$ is the midpoint of $AB$. The line $DF$ meets the diagonal $AC$ at $M$. Prove that $AM : MC = 1 : 2$ and $DM : MF = 2 : 1$. [5 marks]

**19.** Find the components of $\bar{a} = -2\bar{i} - 2\bar{k}$ when the coordinate axes are rotated about the $x$-axis through $90^\circ$. [3 marks]

**20.** A straight line cuts the lines $BC, CA, AB$ of triangle $ABC$ at $D, E, F$. If $\frac{BD}{DC} = -\frac{3}{5}$ and $\frac{CE}{EA} = \frac{2}{3}$ (signed ratios), find $\frac{AF}{FB}$. [1 mark]

## Key sheet

| Q | Answer |
|---|---|
| 1 | $\frac{1}{25}(16\bar{i}+12\bar{j}-15\bar{k})$ |
| 2 | $-4\bar{i}+20\bar{j}+22\bar{k}$ |
| 3 | $\bar{i}-5\bar{j}-7\bar{k}$ |
| 4 | $105^\circ$ |
| 5 | $\sqrt{46}$ |
| 6 | $(\frac{4}{9},-\frac{8}{9},-\frac{1}{9})$ |
| 7 | $p=15$ |
| 8 | $p=-12$ |
| 9 | $\bar{r}=-2\bar{a}+3\bar{b}-3\bar{c}$ ($x=-2$, $y=3$, $z=-3$) |
| 10 | The determinant is $0$, so $A, B, C, D$ are coplanar |
| 11 | $(-3,5,2)$ |
| 12 | $(3,-7,7)$ |
| 13 | $(-4,3,2)$ and $(-4,1,3)$ |
| 14 | $(13,1,\frac{15}{2})$ |
| 15 | $\bar{r}=(1-t)(2\bar{i}+2\bar{j}+2\bar{k})+t(\bar{i}-\bar{j}+5\bar{k})$; $\frac{x-2}{-1}=\frac{y-2}{-3}=\frac{z-2}{3}$ |
| 16 | $\bar{r}=(1-s-t)(\bar{i}-2\bar{k})+s(3\bar{i}+4\bar{j}+3\bar{k})+t(-\bar{i}+3\bar{j}+2\bar{k})$; point $(-1,10,11)$ |
| 17 | $\frac{18}{35}\bar{i}-\frac{51}{35}\bar{j}-\frac{3}{7}\bar{k}$ (or any multiple) |
| 18 | Proved (see the working) |
| 19 | $(-2,-2,0)$ |
| 20 | $\frac{5}{2}$ |

## Solutions

**1.** Find the unit vector in the direction of $\bar{a} = 16\bar{i} + 12\bar{j} - 15\bar{k}$. [1]

$|\bar{a}| = \sqrt{256 + 144 + 225} = \sqrt{625} = 25$

$\text{Unit vector} = \frac{\bar{a}}{|\bar{a}|} = \frac{1}{25}(16\bar{i} + 12\bar{j} - 15\bar{k})$

**Ans:** $\frac{1}{25}(16\bar{i} + 12\bar{j} - 15\bar{k})$

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**2.** If $\bar{a} = -\bar{i} - 3\bar{j} - 3\bar{k}$ and $\bar{b} = -\bar{i} + 13\bar{j} + 14\bar{k}$, find the vector of magnitude $30$ in the direction of $\bar{a} + \bar{b}$. [2]

$\bar{a} + \bar{b} = -2\bar{i} + 10\bar{j} + 11\bar{k}$

$|\bar{a} + \bar{b}| = \sqrt{4 + 100 + 121} = 15$

$\text{Required vector} = \frac{30(\bar{a}+\bar{b})}{|\bar{a}+\bar{b}|} = \frac{30}{15}(-2\bar{i} + 10\bar{j} + 11\bar{k}) = -4\bar{i} + 20\bar{j} + 22\bar{k}$

**Ans:** $-4\bar{i} + 20\bar{j} + 22\bar{k}$

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**3.** If $\overline{OC} = -3\bar{i} + 2\bar{j} + 4\bar{k}$ and $\overline{OD} = -2\bar{i} - 3\bar{j} - 3\bar{k}$, find $\overline{CD}$. [1]

$\overline{CD} = \overline{OD} - \overline{OC}$

$= (-2\bar{i} - 3\bar{j} - 3\bar{k}) - (-3\bar{i} + 2\bar{j} + 4\bar{k}) = \bar{i} - 5\bar{j} - 7\bar{k}$

**Ans:** $\bar{i} - 5\bar{j} - 7\bar{k}$

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**4.** If $(\bar{a}, \bar{b}) = 75°$, find $(5\bar{a}, -2\bar{b})$. [1]

The scalars $5$ and $-2$ have opposite signs.

$(5\bar{a}, -2\bar{b}) = 180° - 75° = 105°$

**Ans:** $105°$

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**5.** Find the distance $|\overline{AB}|$ between $A(-1, -1, 0)$ and $B(0, 5, 3)$. [1]

$\overline{AB} = \bar{i} + 6\bar{j} + 3\bar{k}$

$|\overline{AB}| = \sqrt{1 + 36 + 9} = \sqrt{46}$

**Ans:** $\sqrt{46}$

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**6.** Find the direction cosines of the vector $4\bar{i} - 8\bar{j} - \bar{k}$. [2]

$r = \sqrt{16 + 64 + 1} = 9$

$l = \frac{x}{r},\ m = \frac{y}{r},\ n = \frac{z}{r}$

$\text{Direction cosines} = (\frac{4}{9}, -\frac{8}{9}, -\frac{1}{9})$

**Ans:** $(\frac{4}{9}, -\frac{8}{9}, -\frac{1}{9})$

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**7.** Find $p$ if the vectors $18\bar{i} + p\bar{j} - 12\bar{k}$ and $6\bar{i} + 5\bar{j} - 4\bar{k}$ are collinear. [2]

Collinear vectors have proportional components.

$\frac{-12}{-4} = \frac{p}{5}$

$\Rightarrow p = 15$

**Ans:** $p = 15$

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**8.** Find $p$ so that the points $A(-4,-1,-1)$, $B(-6,2,0)$ and $C(p,11,3)$ are collinear. [3]

$\overline{AB} = (-2,3,1)$

$\overline{AC} = (p+4,12,4)$

For collinear points the components are proportional:

$\frac{12}{3} = \frac{p+4}{-2}$

$\Rightarrow p = -12$

**Ans:** $p = -12$

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**9.** Express $\bar{r} = 3\bar{i} + 5\bar{j} - 7\bar{k}$ as a linear combination of $\bar{a} = 3\bar{i} - \bar{j} - \bar{k}$, $\bar{b} = 2\bar{i} - \bar{k}$ and $\bar{c} = -\bar{i} - \bar{j} + 2\bar{k}$. [5]

Let $3\bar{i} + 5\bar{j} - 7\bar{k} = x(3\bar{i} - \bar{j} - \bar{k}) + y(2\bar{i} - \bar{k}) + z(-\bar{i} - \bar{j} + 2\bar{k})$.

Comparing the coefficients of $\bar{i}, \bar{j}, \bar{k}$:

$3x + 2y - z = 3$

$-x - z = 5$

$-x - y + 2z = -7$

Solving by Cramer's rule:

$$\Delta = \begin{vmatrix} 3 & 2 & -1 \\ -1 & 0 & -1 \\ -1 & -1 & 2 \end{vmatrix} = 2$$

$$\Delta_1 = \begin{vmatrix} 3 & 2 & -1 \\ 5 & 0 & -1 \\ -7 & -1 & 2 \end{vmatrix} = -4, \quad x = \frac{\Delta_1}{\Delta} = -2$$

$$\Delta_2 = \begin{vmatrix} 3 & 3 & -1 \\ -1 & 5 & -1 \\ -1 & -7 & 2 \end{vmatrix} = 6, \quad y = \frac{\Delta_2}{\Delta} = 3$$

$$\Delta_3 = \begin{vmatrix} 3 & 2 & 3 \\ -1 & 0 & 5 \\ -1 & -1 & -7 \end{vmatrix} = -6, \quad z = \frac{\Delta_3}{\Delta} = -3$$

$\therefore \bar{r} = -2\bar{a} + 3\bar{b} - 3\bar{c}$

**Ans:** $\bar{r} = -2\bar{a} + 3\bar{b} - 3\bar{c}$ **($x = -2$, $y = 3$, $z = -3$)**

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**10.** Show that the points $A(3,-2,1)$, $B(5,-3,1)$, $C(5,-5,4)$ and $D(5,-1,-2)$ are coplanar. [5]

$\overline{AB} = 2\bar{i} - \bar{j}$, $\overline{AC} = 2\bar{i} - 3\bar{j} + 3\bar{k}$, $\overline{AD} = 2\bar{i} + \bar{j} - 3\bar{k}$

The four points are coplanar when these three vectors are coplanar (linearly dependent).

$$\begin{vmatrix} 2 & -1 & 0 \\ 2 & -3 & 3 \\ 2 & 1 & -3 \end{vmatrix} = 2(6) - (-1)(-12) + 0(8)$$

$= 0$

The determinant is zero, so $\overline{AB}, \overline{AC}, \overline{AD}$ are coplanar and $A, B, C, D$ are coplanar.

**Ans: The determinant is $0$, so $A, B, C, D$ are coplanar**

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**11.** The midpoint of $AB$ is $(-\frac{1}{2}, 5, \frac{7}{2})$ and $A = (2,5,5)$. Find $B$. [1]

$\frac{\bar{a}+\bar{b}}{2} = \bar{m} \Rightarrow \bar{b} = 2\bar{m} - \bar{a}$

$= (-1, 10, 7) - (2,5,5) = (-3,5,2)$

**Ans:** $(-3,5,2)$

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**12.** Find the point that divides the segment joining $A(5,-3,-1)$ and $B(6,-1,-5)$ externally in the ratio $2:3$. [2]

$\bar{r} = \frac{m\bar{b}-n\bar{a}}{m-n}$
$= \frac{2(6,-1,-5)-3(5,-3,-1)}{2-3}$
$= \frac{(-3,7,-7)}{-1} = (3,-7,7)$

**Ans:** $(3,-7,7)$

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**13.** Find the points of trisection of the segment joining $A(-4,5,1)$ and $B(-4,-1,4)$. [3]

The points of trisection divide $AB$ internally in the ratios $1:2$ and $2:1$.
$\frac{1(-4,-1,4)+2(-4,5,1)}{3} = \frac{(-12,9,6)}{3} = (-4,3,2)$
$\frac{2(-4,-1,4)+1(-4,5,1)}{3} = \frac{(-12,3,9)}{3} = (-4,1,3)$

**Ans:** $(-4,3,2)$ **and** $(-4,1,3)$

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**14.** Find the incentre of the triangle whose vertices are $A(22,1,3)$, $B(13,1,15)$ and $C(4,1,3)$. [5]

$a = BC = \sqrt{81+0+144} = 15$
$b = CA = \sqrt{324+0+0} = 18$
$c = AB = \sqrt{81+0+144} = 15$
Incentre $I = \frac{a\bar{a}+b\bar{b}+c\bar{c}}{a+b+c}$
$= \frac{15(22,1,3)+18(13,1,15)+15(4,1,3)}{48}$
$= \frac{(624,48,360)}{48} = (13,1,\frac{15}{2})$

**Ans:** $(13,1,\frac{15}{2})$

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**15.** Find the vector and cartesian equations of the line through $A(2,2,2)$ and $B(1,-1,5)$. [3]

$\bar{r} = (1-t)\bar{a} + t\bar{b} = (1-t)(2\bar{i}+2\bar{j}+2\bar{k}) + t(\bar{i}-\bar{j}+5\bar{k})$
$= (2-t)\bar{i} + (2-3t)\bar{j} + (2+3t)\bar{k}$
Direction $\overline{AB} = (-1,-3,3)$
$\frac{x-2}{-1} = \frac{y-2}{-3} = \frac{z-2}{3}$

**Ans:** $\bar{r} = (1-t)(2\bar{i}+2\bar{j}+2\bar{k}) + t(\bar{i}-\bar{j}+5\bar{k})$; $\frac{x-2}{-1} = \frac{y-2}{-3} = \frac{z-2}{3}$

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**16.** Write the vector equation of the plane through the points $A(1,0,-2)$, $B(3,4,3)$ and $C(-1,3,2)$, and find its point with $s=1$, $t=2$. [2]

$\bar{r} = (1-s-t)\bar{a} + s\bar{b} + t\bar{c}$
$\bar{r} = (1-s-t)(\bar{i}-2\bar{k}) + s(3\bar{i}+4\bar{j}+3\bar{k}) + t(-\bar{i}+3\bar{j}+2\bar{k})$
$s=1,\ t=2:\ \bar{r} = -2(1,0,-2) + 1(3,4,3) + 2(-1,3,2) = (-1,10,11)$

**Ans:** $\bar{r} = (1-s-t)(\bar{i}-2\bar{k}) + s(3\bar{i}+4\bar{j}+3\bar{k}) + t(-\bar{i}+3\bar{j}+2\bar{k})$; **point** $(-1,10,11)$

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**17.** Find a vector along the external bisector of the angle between $\bar{a} = -2\bar{i}-6\bar{j}-3\bar{k}$ and $\bar{b} = -4\bar{i}+3\bar{j}$. [2]

$|\bar{a}| = \sqrt{4+36+9} = 7, \quad |\bar{b}| = \sqrt{16+9+0} = 5$
$\frac{\bar{a}}{|\bar{a}|} - \frac{\bar{b}}{|\bar{b}|} = \frac{1}{7}(-2\bar{i}-6\bar{j}-3\bar{k}) - \frac{1}{5}(-4\bar{i}+3\bar{j})$
$= \frac{18}{35}\bar{i} - \frac{51}{35}\bar{j} - \frac{3}{7}\bar{k}$

Any non-zero multiple of this vector is also along the external bisector.

**Ans:** $\frac{18}{35}\bar{i} - \frac{51}{35}\bar{j} - \frac{3}{7}\bar{k}$ **(or any multiple)**

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**18.** $ABCD$ is a parallelogram and $F$ is the midpoint of $AB$. The line $DF$ meets the diagonal $AC$ at $M$. Prove that $AM : MC = 1 : 2$ and $DM : MF = 2 : 1$. [5]

Take $A$ as the origin with $\overrightarrow{AB} = \bar{b}$ and $\overrightarrow{AD} = \bar{d}$. Then $\overrightarrow{AC} = \bar{b} + \bar{d}$ and $\overrightarrow{AF} = \frac{\bar{b}}{2}$.

The point dividing $DF$ in the ratio $2 : 1$ has position vector $\frac{2 \cdot \frac{\bar{b}}{2} + 1 \cdot \bar{d}}{3} = \frac{\bar{b} + \bar{d}}{3}$

The point dividing $AC$ in the ratio $1 : 2$ has position vector $\frac{1(\bar{b} + \bar{d}) + 2 \cdot \bar{0}}{3} = \frac{\bar{b} + \bar{d}}{3}$

So this one point lies on both $DF$ and $AC$; it is their meeting point $M$.

$\therefore AM : MC = 1 : 2$ and $DM : MF = 2 : 1$.

**Ans: Proved (see the working)**

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**19.** Find the components of $\bar{a} = -2\bar{i} - 2\bar{k}$ when the coordinate axes are rotated about the $x$-axis through $90^\circ$. [3]

About the $x$-axis through $\alpha$ the new components are $(a_1,\ a_2\cos\alpha + a_3\sin\alpha,\ -a_2\sin\alpha + a_3\cos\alpha)$.

$\cos 90^\circ = 0, \quad \sin 90^\circ = 1$

Substituting $a_1 = -2$, $a_2 = 0$, $a_3 = -2$:

$((-2),\ 0 \cdot 0 + (-2) \cdot 1,\ -0 \cdot 1 + (-2) \cdot 0)$

$= (-2, -2, 0)$

**Ans:** $(-2, -2, 0)$

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**20.** A straight line cuts the lines $BC, CA, AB$ of triangle $ABC$ at $D, E, F$. If $\frac{BD}{DC} = -\frac{3}{5}$ and $\frac{CE}{EA} = \frac{2}{3}$ (signed ratios), find $\frac{AF}{FB}$. [1]

By Menelaus' theorem $\frac{BD}{DC} \cdot \frac{CE}{EA} \cdot \frac{AF}{FB} = -1$

$\frac{AF}{FB} = \frac{-1}{(-\frac{3}{5})(\frac{2}{3})} = \frac{5}{2}$

**Ans:** $\frac{5}{2}$

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