EXPRESSWALA ACADEMY

Maths 1A: Addition of Vectors

Question Paper, Set 03

Time: 1 hour 30 minutes | Maximum marks: 50

Name: _____ Class: _____ Date: _____

1. Answer all 20 questions. The marks for each question are shown in brackets.
2. Show your working for the 2, 3 and 5 mark questions.
3. The key sheet and worked solutions are at the end of this paper.

1. Find the unit vector along $4\bar{i} - 4\bar{j}$. [1 mark]

2. Find the vectors of magnitude 34 that are parallel to $\bar{a} = 8\bar{i} - 9\bar{j} + 12\bar{k}$. [2 marks]

3. The position vectors of L and M are $\bar{i} - 2\bar{j} - 2\bar{k}$ and $-3\bar{i} - \bar{j} + 5\bar{k}$. Find $\overline{LM}$. [1 mark]

4. If $(\bar{a}, \bar{b}) = 75^\circ$, find $(-4\bar{a}, -4\bar{b})$. [1 mark]

5. Find the magnitude of $\overline{OP}$, where $P = (-6, 4, 1)$. [1 mark]

6. Find the direction cosines of $\overline{AB}$, where $A = (-3, -3, -4)$ and $B = (-5, -1, -3)$. [2 marks]

7. Find p if the vectors $8\bar{i} - 12\bar{j} + p\bar{k}$ and $-4\bar{i} + 6\bar{j} + \bar{k}$ are collinear. [2 marks]

8. Show that the points $A(2, 4, 1)$, $B(-1, 1, 2)$ and $C(8, 10, -1)$ are collinear. [3 marks]

9. Write $\bar{r} = -9\bar{i} + 3\bar{j} + \bar{k}$ in terms of the non-coplanar vectors $\bar{a} = 3\bar{i} - \bar{j}$, $\bar{b} = -2\bar{i} + \bar{j} + 3\bar{k}$ and $\bar{c} = -2\bar{i} + \bar{j} + 2\bar{k}$. [5 marks]

10. Find p so that the vectors $-3\bar{i} + \bar{j} + 3\bar{k}$, $-2\bar{i} + 3\bar{k}$ and $p\bar{i} - 2\bar{j} - 12\bar{k}$ are linearly dependent. [5 marks]

11. Find the midpoint of the segment joining $A(7, 5, 0)$ and $B(-4, 5, 6)$. [1 mark]

12. Find the point that divides the segment joining $A(6, -2, 2)$ and $B(2, -2, 2)$ internally in the ratio $3 : 1$. [2 marks]

13. The point $C(-3, -2, 3)$ divides AB internally in the ratio $2 : 1$, and $A = (1, 4, 1)$. Find B. [3 marks]

14. Find the incentre of the triangle whose vertices are $A(8, 4, 1)$, $B(0, 4, 1)$ and $C(0, 4, 7)$. [5 marks]

15. Find the vector and cartesian equations of the line through the origin and the point $P(4, 1, 4)$. [3 marks]

16. Write the vector equation of the plane through the points $3\bar{i} + \bar{j} + \bar{k}$ and $-3\bar{i} + \bar{j} - 3\bar{k}$ parallel to $\bar{i} - \bar{j} - 2\bar{k}$, and find its point with $s = 1$, $t = 1$. [2 marks]

17. Find a vector along the internal bisector of the angle between $\bar{a} = 19\bar{i} - 4\bar{j} - 8\bar{k}$ and $\bar{b} = -4\bar{i} + 8\bar{j} + \bar{k}$. [2 marks]

18. $ABCD$ is a parallelogram whose diagonals meet at E. For any point O, prove that $\overline{OA} + \overline{OB} + \overline{OC} + \overline{OD} = 4\,\overline{OE}$. [5 marks]

19. Find the components of $\bar{a} = 3\bar{i} - 3\bar{j} - 4\bar{k}$ when the coordinate axes are rotated about the z-axis through 60°. [3 marks]

20. In triangle ABC, the lines joining a point P to A, B, C meet BC, CA, AB at D, E, F. If $\frac{BD}{DC} = \frac{3}{5}$ and $\frac{CE}{EA} = 4$, find $\frac{AF}{FB}$. [1 mark]

Key sheet

Q	Answer
1	$\frac{1}{\sqrt{32}}(4\bar{i}-4\bar{j})$
2	$\pm(16\bar{i}-18\bar{j}+24\bar{k})$
3	$-4\bar{i}+\bar{j}+7\bar{k}$
4	75°
5	$\sqrt{53}$
6	$(-\frac{2}{3},\frac{2}{3},\frac{1}{3})$
7	$p=-2$
8	$\overline{AC}=-2\,\overline{AB}$, so A, B, C are collinear
9	$\bar{r}=-3\bar{a}+\bar{b}-\bar{c}$ ($x=-3$, $y=1$, $z=-1$)
10	$p=10$
11	$(\frac{3}{2},5,3)$
12	$(3,-2,2)$
13	$B=(-5,-5,4)$
14	$(2,4,3)$
15	$\bar{r}=t(4\bar{i}+\bar{j}+4\bar{k})$; $\frac{x}{4}=\frac{y}{1}=\frac{z}{4}$
16	$\bar{r}=(1-s)(3\bar{i}+\bar{j}+\bar{k})+s(-3\bar{i}+\bar{j}-3\bar{k})+t(\bar{i}-\bar{j}-2\bar{k})$; point $(-2,0,-5)$
17	$\frac{29}{63}\bar{i}+\frac{44}{63}\bar{j}-\frac{17}{63}\bar{k}$ (or any multiple)
18	Proved (see the working)
19	$(\frac{3}{2}-\frac{3\sqrt{3}}{2},-\frac{3\sqrt{3}}{2}-\frac{3}{2},-4)$
20	$\frac{5}{12}$

Solutions

1. Find the unit vector along $4\bar{i} - 4\bar{j}$. [1]

$|4\bar{i} - 4\bar{j}| = \sqrt{16 + 16 + 0} = \sqrt{32}$

$\text{Unit vector} = \frac{1}{\sqrt{32}}(4\bar{i} - 4\bar{j})$

Ans: $\frac{1}{\sqrt{32}}(4\bar{i} - 4\bar{j})$

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2. Find the vectors of magnitude 34 that are parallel to $\bar{a} = 8\bar{i} - 9\bar{j} + 12\bar{k}$. [2]

$|\bar{a}| = \sqrt{64 + 81 + 144} = 17$

$\text{Required vectors} = \pm\frac{34\,\bar{a}}{|\bar{a}|} = \pm\frac{34}{17}(8\bar{i} - 9\bar{j} + 12\bar{k})$

$= \pm(16\bar{i} - 18\bar{j} + 24\bar{k})$

Ans: $\pm(16\bar{i} - 18\bar{j} + 24\bar{k})$

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3. The position vectors of L and M are $\bar{i} - 2\bar{j} - 2\bar{k}$ and $-3\bar{i} - \bar{j} + 5\bar{k}$. Find $\overline{LM}$. [1]

$\overline{LM} = \overline{OM} - \overline{OL}$

$= (-3\bar{i} - \bar{j} + 5\bar{k}) - (\bar{i} - 2\bar{j} - 2\bar{k}) = -4\bar{i} + \bar{j} + 7\bar{k}$

Ans: $-4\bar{i} + \bar{j} + 7\bar{k}$

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4. If $(\bar{a}, \bar{b}) = 75^\circ$, find $(-4\bar{a}, -4\bar{b})$. [1]

The scalars -4 and -4 have the same sign.

$(-4\bar{a}, -4\bar{b}) = (\bar{a}, \bar{b}) = 75^\circ$

Ans: 75°

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5. Find the magnitude of $\overline{OP}$, where $P = (-6, 4, 1)$. [1]

$\overline{OP} = -6\bar{i} + 4\bar{j} + \bar{k}$

$|\overline{OP}| = \sqrt{36 + 16 + 1} = \sqrt{53}$

Ans: $\sqrt{53}$

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6. Find the direction cosines of $\overline{AB}$, where $A = (-3, -3, -4)$ and $B = (-5, -1, -3)$. [2]

$\overline{AB} = -2\bar{i} + 2\bar{j} + \bar{k}$

$|\overline{AB}| = \sqrt{4 + 4 + 1} = 3$

$\text{Direction cosines} = (-\frac{2}{3}, \frac{2}{3}, \frac{1}{3})$

Ans: $(-\frac{2}{3}, \frac{2}{3}, \frac{1}{3})$

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7. Find p if the vectors $8\bar{i} - 12\bar{j} + p\bar{k}$ and $-4\bar{i} + 6\bar{j} + \bar{k}$ are collinear. [2]

Collinear vectors have proportional components.

$\frac{8}{-4} = \frac{p}{1}$

$\Rightarrow p = -2$

Ans: $p = -2$

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8. Show that the points $A(2,4,1)$, $B(-1,1,2)$ and $C(8,10,-1)$ are collinear. [3]

$\overline{AB} = \overline{OB} - \overline{OA} = -3\bar{i} - 3\bar{j} + \bar{k}$

$\overline{AC} = \overline{OC} - \overline{OA} = 6\bar{i} + 6\bar{j} - 2\bar{k}$

$\overline{AC} = -2\,\overline{AB}$

$\overline{AB}$ and $\overline{AC}$ are parallel and have the common point A, so A, B, C are collinear.

Ans: $\overline{AC} = -2\,\overline{AB}$**, so** A, B, C **are collinear**

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9. Write $\bar{r} = -9\bar{i} + 3\bar{j} + \bar{k}$ in terms of the non-coplanar vectors $\bar{a} = 3\bar{i} - \bar{j}$, $\bar{b} = -2\bar{i} + \bar{j} + 3\bar{k}$ and $\bar{c} = -2\bar{i} + \bar{j} + 2\bar{k}$. [5]

Let $-9\bar{i} + 3\bar{j} + \bar{k} = x(3\bar{i} - \bar{j}) + y(-2\bar{i} + \bar{j} + 3\bar{k}) + z(-2\bar{i} + \bar{j} + 2\bar{k})$.

Comparing the coefficients of $\bar{i}, \bar{j}, \bar{k}$:

$3x - 2y - 2z = -9$

$-x + y + z = 3$

$3y + 2z = 1$

Solving by Cramer's rule:

$$\Delta = \begin{vmatrix} 3 & -2 & -2 \\ -1 & 1 & 1 \\ 0 & 3 & 2 \end{vmatrix} = -1$$

$$\Delta_1 = \begin{vmatrix} -9 & -2 & -2 \\ 3 & 1 & 1 \\ 1 & 3 & 2 \end{vmatrix} = 3, \quad x = \frac{\Delta_1}{\Delta} = -3$$

$$\Delta_2 = \begin{vmatrix} 3 & -9 & -2 \\ -1 & 3 & 1 \\ 0 & 1 & 2 \end{vmatrix} = -1, \quad y = \frac{\Delta_2}{\Delta} = 1$$

$$\Delta_3 = \begin{vmatrix} 3 & -2 & -9 \\ -1 & 1 & 3 \\ 0 & 3 & 1 \end{vmatrix} = 1, \quad z = \frac{\Delta_3}{\Delta} = -1$$

$\therefore \bar{r} = -3\bar{a} + \bar{b} - \bar{c}$

Ans: $\bar{r} = -3\bar{a} + \bar{b} - \bar{c}$ **(**$x = -3$**,** $y = 1$**,** $z = -1$**)**

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10. Find p so that the vectors $-3\bar{i} + \bar{j} + 3\bar{k}$, $-2\bar{i} + 3\bar{k}$ and $p\bar{i} - 2\bar{j} - 12\bar{k}$ are linearly dependent. [5]

Three vectors are linearly dependent (coplanar) when the determinant of their components is zero.

$$\begin{vmatrix} -3 & 1 & 3 \\ -2 & 0 & 3 \\ p & -2 & -12 \end{vmatrix} = 0$$

$\Rightarrow 3p - 30 = 0$

$\Rightarrow p = 10$

Ans: $p = 10$

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11. Find the midpoint of the segment joining $A(7,5,0)$ and $B(-4,5,6)$. [1]

$\text{Midpoint} = \frac{\bar{a}+\bar{b}}{2} = \frac{(3,10,6)}{2}$

$= (\frac{3}{2}, 5, 3)$

Ans: $(\frac{3}{2}, 5, 3)$

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12. Find the point that divides the segment joining $A(6,-2,2)$ and $B(2,-2,2)$ internally in the ratio $3:1$. [2]

$\bar{r} = \frac{m\bar{b}+n\bar{a}}{m+n}$
$= \frac{3(2,-2,2)+1(6,-2,2)}{3+1}$
$= \frac{(12,-8,8)}{4} = (3,-2,2)$

Ans: $(3,-2,2)$

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13. The point $C(-3,-2,3)$ divides AB internally in the ratio $2:1$, and $A=(1,4,1)$. Find B. [3]

$\bar{c} = \frac{2\bar{b}+1\bar{a}}{3} \Rightarrow \bar{b} = \frac{3\bar{c}-1\bar{a}}{2}$
$= \frac{(-9,-6,9)-(1,4,1)}{2}$
$= (-5,-5,4)$

Ans: $B=(-5,-5,4)$

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14. Find the incentre of the triangle whose vertices are $A(8,4,1)$, $B(0,4,1)$ and $C(0,4,7)$. [5]

$a = BC = \sqrt{0+0+36} = 6$
$b = CA = \sqrt{64+0+36} = 10$
$c = AB = \sqrt{64+0+0} = 8$
$\text{Incentre } I = \frac{a\bar{a}+b\bar{b}+c\bar{c}}{a+b+c}$
$= \frac{6(8,4,1)+10(0,4,1)+8(0,4,7)}{24}$
$= \frac{(48,96,72)}{24} = (2,4,3)$

Ans: $(2,4,3)$

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15. Find the vector and cartesian equations of the line through the origin and the point $P(4,1,4)$. [3]

$\bar{r} = t\,\overline{OP} = t(4\bar{i}+\bar{j}+4\bar{k})$
$\text{Cartesian: } \frac{x}{4} = \frac{y}{1} = \frac{z}{4}$

Ans: $\bar{r} = t(4\bar{i}+\bar{j}+4\bar{k})$; $\frac{x}{4} = \frac{y}{1} = \frac{z}{4}$

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16. Write the vector equation of the plane through the points $3\bar{i}+\bar{j}+\bar{k}$ and $-3\bar{i}+\bar{j}-3\bar{k}$ parallel to $\bar{i}-\bar{j}-2\bar{k}$, and find its point with $s=1$, $t=1$. [2]

$\bar{r} = (1-s)\bar{a} + s\bar{b} + t\bar{c}$
$\bar{r} = (1-s)(3\bar{i}+\bar{j}+\bar{k}) + s(-3\bar{i}+\bar{j}-3\bar{k}) + t(\bar{i}-\bar{j}-2\bar{k})$
$s=1,\ t=1:\ \bar{r} = (-2,0,-5)$

Ans: $\bar{r} = (1-s)(3\bar{i}+\bar{j}+\bar{k}) + s(-3\bar{i}+\bar{j}-3\bar{k}) + t(\bar{i}-\bar{j}-2\bar{k})$; **point** $(-2,0,-5)$

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17. Find a vector along the internal bisector of the angle between $\bar{a} = 19\bar{i}-4\bar{j}-8\bar{k}$ and $\bar{b} = -4\bar{i}+8\bar{j}+\bar{k}$. [2]

$|\bar{a}| = \sqrt{361+16+64} = 21, \quad |\bar{b}| = \sqrt{16+64+1} = 9$
$\frac{\bar{a}}{|\bar{a}|} + \frac{\bar{b}}{|\bar{b}|} = \frac{1}{21}(19\bar{i}-4\bar{j}-8\bar{k}) + \frac{1}{9}(-4\bar{i}+8\bar{j}+\bar{k})$
$= \frac{29}{63}\bar{i} + \frac{44}{63}\bar{j} - \frac{17}{63}\bar{k}$

Any non-zero multiple of this vector is also along the internal bisector.

Ans: $\frac{29}{63}\bar{i} + \frac{44}{63}\bar{j} - \frac{17}{63}\bar{k}$ **(or any multiple)**

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18. $ABCD$ is a parallelogram whose diagonals meet at E. For any point O, prove that $\overline{OA}+\overline{OB}+\overline{OC}+\overline{OD}=4\,\overline{OE}$. [5]

The diagonals of a parallelogram bisect each other, so E is the midpoint of AC and of BD.
$\overline{OE}=\frac{\overline{OA}+\overline{OC}}{2}\Rightarrow \overline{OA}+\overline{OC}=2\,\overline{OE}$
$\overline{OE}=\frac{\overline{OB}+\overline{OD}}{2}\Rightarrow \overline{OB}+\overline{OD}=2\,\overline{OE}$
Adding:
$\overline{OA}+\overline{OB}+\overline{OC}+\overline{OD}=4\,\overline{OE}$

Ans: Proved (see the working)

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19. Find the components of $\bar{a}=3\bar{i}-3\bar{j}-4\bar{k}$ when the coordinate axes are rotated about the z-axis through 60°. [3]

About the z-axis through α the new components are $(a_1\cos\alpha+a_2\sin\alpha,\ -a_1\sin\alpha+a_2\cos\alpha,\ a_3)$.
$\cos 60^\circ=\frac{1}{2},\quad \sin 60^\circ=\frac{\sqrt{3}}{2}$
Substituting $a_1=3$, $a_2=-3$, $a_3=-4$:
$(3\cdot\frac{1}{2}+(-3)\cdot\frac{\sqrt{3}}{2},\ -3\cdot\frac{\sqrt{3}}{2}+(-3)\cdot\frac{1}{2},\ (-4))$
$=(\frac{3}{2}-\frac{3\sqrt{3}}{2},-\frac{3\sqrt{3}}{2}-\frac{3}{2},-4)$
Ans: $(\frac{3}{2}-\frac{3\sqrt{3}}{2},-\frac{3\sqrt{3}}{2}-\frac{3}{2},-4)$

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20. In triangle ABC, the lines joining a point P to A,B,C meet BC,CA,AB at D,E,F. If $\frac{BD}{DC}=\frac{3}{5}$ and $\frac{CE}{EA}=4$, find $\frac{AF}{FB}$. [1]

By Ceva's theorem $\frac{BD}{DC}\cdot\frac{CE}{EA}\cdot\frac{AF}{FB}=1$
$\frac{AF}{FB}=\frac{1}{\frac{3}{5}\times 4}=\frac{5}{12}$
Ans: $\frac{5}{12}$
