EXPRESSWALA ACADEMY

Maths 1A: Addition of Vectors

Question Paper, Set 10

Time: 1 hour 30 minutes | Maximum marks: 50

Name: _____ Class: _____ Date: _____

1. Answer all 20 questions. The marks for each question are shown in brackets.
2. Show your working for the 2, 3 and 5 mark questions.
3. The key sheet and worked solutions are at the end of this paper.

1. Find the unit vector in the direction of $\bar{a} = -4\bar{i} + 8\bar{j} + \bar{k}$. [1 mark]

2. If $\bar{a} = \bar{i} + 3\bar{j} - 2\bar{k}$ and $\bar{b} = -7\bar{i} + 4\bar{j} - 4\bar{k}$, find the vector of magnitude 11 in the direction of $\bar{a} + \bar{b}$. [2 marks]

3. If $\overline{OP} = -2\bar{i} + 4\bar{j} + 3\bar{k}$ and $\overline{OQ} = -4\bar{i} + 4\bar{j} - 2\bar{k}$, find $\overline{PQ}$. [1 mark]

4. If $(\bar{a}, \bar{b}) = 80^\circ$, find $(-\bar{a}, 4\bar{b})$. [1 mark]

5. Find the distance $|\overline{AB}|$ between $A(5, 1, -1)$ and $B(3, -2, -4)$. [1 mark]

6. Find the direction cosines of the vector $-12\bar{i} + 4\bar{j} + 3\bar{k}$. [2 marks]

7. Find p if the vectors $4\bar{i} - 12\bar{j} + p\bar{k}$ and $-2\bar{i} + 6\bar{j} - \bar{k}$ are collinear. [2 marks]

8. Find p so that the points $A(1, 3, -4)$, $B(4, 0, -7)$ and $C(-8, p, 5)$ are collinear. [3 marks]

9. Express $\bar{r} = \bar{i} + 10\bar{j} + 5\bar{k}$ as a linear combination of $\bar{a} = -2\bar{i} - 2\bar{j} - \bar{k}$, $\bar{b} = -2\bar{i} - \bar{j} + \bar{k}$ and $\bar{c} = \bar{i} + 3\bar{j}$. [5 marks]

10. Show that the points $A(-2, 4, -2)$, $B(0, 6, -4)$, $C(1, 4, 1)$ and $D(-1, 2, 3)$ are coplanar. [5 marks]

11. The midpoint of AB is $(2, -\frac{3}{2}, 1)$ and $A = (1, 1, 7)$. Find B. [1 mark]

12. Find the point that divides the segment joining $A(-3, -1, 2)$ and $B(3, -5, 3)$ externally in the ratio $3 : 2$. [2 marks]

13. Find the points of trisection of the segment joining $A(5, 0, -3)$ and $B(11, -6, -3)$. [3 marks]

14. Find the incentre of the triangle whose vertices are $A(-2, 18, 2)$, $B(-2, 2, 2)$ and $C(-2, 10, 8)$. [5 marks]

15. Find the vector and cartesian equations of the line through $A(4, 2, 0)$ and $B(1, 5, 3)$. [3 marks]

16. Write the vector equation of the plane through the points $A(4, 1, 1)$, $B(3, 2, 1)$ and $C(-3, -3, -2)$, and find its point with $s = 1$, $t = 2$. [2 marks]

17. Find a vector along the external bisector of the angle between $\bar{a} = 8\bar{i} - 15\bar{k}$ and $\bar{b} = 7\bar{i} + 4\bar{j} + 4\bar{k}$. [2 marks]

18. In triangle ABC, S is the circumcentre and O the orthocentre, and $\overline{SA} + \overline{SB} + \overline{SC} = \overline{SO}$. Prove that $\overline{OA} + \overline{OB} + \overline{OC} = 2\,\overline{OS}$. [5 marks]

19. Find the components of $\bar{a} = -\bar{i} + 2\bar{j} + 5\bar{k}$ when the coordinate axes are rotated about the x-axis through 60°. [3 marks]

20. A straight line cuts the lines BC, CA, AB of triangle ABC at D, E, F. If $\frac{BD}{DC} = -\frac{5}{2}$ and $\frac{CE}{EA} = \frac{2}{3}$ (signed ratios), find $\frac{AF}{FB}$. [1 mark]

Key sheet

Q	Answer
1	$\frac{1}{9}(-4\bar{i}+8\bar{j}+\bar{k})$
2	$-6\bar{i}+7\bar{j}-6\bar{k}$
3	$-2\bar{i}-5\bar{k}$
4	100°
5	$\sqrt{22}$
6	$(-\frac{12}{13}, \frac{4}{13}, \frac{3}{13})$
7	$p=2$
8	$p=12$
9	$\bar{r}=-2\bar{a}+3\bar{b}+3\bar{c}$ ($x=-2$, $y=3$, $z=3$)
10	The determinant is 0, so A, B, C, D are coplanar
11	$(3,-4,-5)$
12	$(15,-13,5)$
13	$(7,-2,-3)$ and $(9,-4,-3)$
14	$(-2, 10, \frac{14}{3})$
15	$\bar{r}=(1-t)(4\bar{i}+2\bar{j})+t(\bar{i}+5\bar{j}+3\bar{k})$; $\frac{x-4}{-3}=\frac{y-2}{3}=\frac{z}{3}$
16	$\bar{r}=(1-s-t)(4\bar{i}+\bar{j}+\bar{k})+s(3\bar{i}+2\bar{j}+\bar{k})+t(-3\bar{i}-3\bar{j}-2\bar{k})$; point $(-11,-6,-5)$
17	$-\frac{47}{153}\bar{i}-\frac{4}{9}\bar{j}-\frac{203}{153}\bar{k}$ (or any multiple)
18	Proved (see the working)
19	$(-1, 1+\frac{5\sqrt{3}}{2}, \frac{5}{2}-\sqrt{3})$
20	$\frac{3}{5}$

Solutions

1. Find the unit vector in the direction of $\bar{a} = -4\bar{i} + 8\bar{j} + \bar{k}$. [1]

$|\bar{a}| = \sqrt{16 + 64 + 1} = \sqrt{81} = 9$

$\text{Unit vector} = \frac{\bar{a}}{|\bar{a}|} = \frac{1}{9}(-4\bar{i} + 8\bar{j} + \bar{k})$

Ans: $\frac{1}{9}(-4\bar{i} + 8\bar{j} + \bar{k})$

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2. If $\bar{a} = \bar{i} + 3\bar{j} - 2\bar{k}$ and $\bar{b} = -7\bar{i} + 4\bar{j} - 4\bar{k}$, find the vector of magnitude 11 in the direction of $\bar{a} + \bar{b}$. [2]

$\bar{a} + \bar{b} = -6\bar{i} + 7\bar{j} - 6\bar{k}$

$|\bar{a} + \bar{b}| = \sqrt{36 + 49 + 36} = 11$

$\text{Required vector} = \frac{11(\bar{a}+\bar{b})}{|\bar{a}+\bar{b}|} = \frac{11}{11}(-6\bar{i} + 7\bar{j} - 6\bar{k}) = -6\bar{i} + 7\bar{j} - 6\bar{k}$

Ans: $-6\bar{i} + 7\bar{j} - 6\bar{k}$

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3. If $\overline{OP} = -2\bar{i} + 4\bar{j} + 3\bar{k}$ and $\overline{OQ} = -4\bar{i} + 4\bar{j} - 2\bar{k}$, find $\overline{PQ}$. [1]

$\overline{PQ} = \overline{OQ} - \overline{OP}$

$= (-4\bar{i} + 4\bar{j} - 2\bar{k}) - (-2\bar{i} + 4\bar{j} + 3\bar{k}) = -2\bar{i} - 5\bar{k}$

Ans: $-2\bar{i} - 5\bar{k}$

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4. If $(\bar{a}, \bar{b}) = 80°$, find $(-\bar{a}, 4\bar{b})$. [1]

The scalars -1 and 4 have opposite signs.

$(-\bar{a}, 4\bar{b}) = 180° - 80° = 100°$

Ans: $100°$

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5. Find the distance $|\overline{AB}|$ between $A(5, 1, -1)$ and $B(3, -2, -4)$. [1]

$\overline{AB} = -2\bar{i} - 3\bar{j} - 3\bar{k}$

$|\overline{AB}| = \sqrt{4 + 9 + 9} = \sqrt{22}$

Ans: $\sqrt{22}$

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6. Find the direction cosines of the vector $-12\bar{i} + 4\bar{j} + 3\bar{k}$. [2]

$r = \sqrt{144 + 16 + 9} = 13$

$l = \frac{x}{r},\ m = \frac{y}{r},\ n = \frac{z}{r}$

$\text{Direction cosines} = (-\frac{12}{13}, \frac{4}{13}, \frac{3}{13})$

Ans: $(-\frac{12}{13}, \frac{4}{13}, \frac{3}{13})$

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7. Find p if the vectors $4\bar{i} - 12\bar{j} + p\bar{k}$ and $-2\bar{i} + 6\bar{j} - \bar{k}$ are collinear. [2]

Collinear vectors have proportional components.

$\frac{4}{-2} = \frac{p}{-1}$

$\Rightarrow p = 2$

Ans: $p = 2$

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8. Find p so that the points $A(1,3,-4)$, $B(4,0,-7)$ and $C(-8,p,5)$ are collinear. [3]

$\overrightarrow{AB} = (3,-3,-3)$

$\overrightarrow{AC} = (-9, p-3, 9)$

For collinear points the components are proportional:

$\frac{-9}{3} = \frac{p-3}{-3}$

$\Rightarrow p = 12$

Ans: $p = 12$

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9. Express $\bar{r} = \bar{i} + 10\bar{j} + 5\bar{k}$ as a linear combination of $\bar{a} = -2\bar{i} - 2\bar{j} - \bar{k}$, $\bar{b} = -2\bar{i} - \bar{j} + \bar{k}$ and $\bar{c} = \bar{i} + 3\bar{j}$. [5]

Let $\bar{i} + 10\bar{j} + 5\bar{k} = x(-2\bar{i} - 2\bar{j} - \bar{k}) + y(-2\bar{i} - \bar{j} + \bar{k}) + z(\bar{i} + 3\bar{j})$.

Comparing the coefficients of $\bar{i}, \bar{j}, \bar{k}$:

$-2x - 2y + z = 1$

$-2x - y + 3z = 10$

$-x + y = 5$

Solving by Cramer's rule:

$$\Delta = \begin{vmatrix} -2 & -2 & 1 \\ -2 & -1 & 3 \\ -1 & 1 & 0 \end{vmatrix} = 9$$

$$\Delta_1 = \begin{vmatrix} 1 & -2 & 1 \\ 10 & -1 & 3 \\ 5 & 1 & 0 \end{vmatrix} = -18, \quad x = \frac{\Delta_1}{\Delta} = -2$$

$$\Delta_2 = \begin{vmatrix} -2 & 1 & 1 \\ -2 & 10 & 3 \\ -1 & 5 & 0 \end{vmatrix} = 27, \quad y = \frac{\Delta_2}{\Delta} = 3$$

$$\Delta_3 = \begin{vmatrix} -2 & -2 & 1 \\ -2 & -1 & 10 \\ -1 & 1 & 5 \end{vmatrix} = 27, \quad z = \frac{\Delta_3}{\Delta} = 3$$

$\therefore \bar{r} = -2\bar{a} + 3\bar{b} + 3\bar{c}$

Ans: $\bar{r} = -2\bar{a} + 3\bar{b} + 3\bar{c}$ **(**$x = -2$**,** $y = 3$**,** $z = 3$**)**

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10. Show that the points $A(-2,4,-2)$, $B(0,6,-4)$, $C(1,4,1)$ and $D(-1,2,3)$ are coplanar. [5]

$\overrightarrow{AB} = 2\bar{i} + 2\bar{j} - 2\bar{k}$, $\overrightarrow{AC} = 3\bar{i} + 3\bar{k}$, $\overrightarrow{AD} = \bar{i} - 2\bar{j} + 5\bar{k}$

The four points are coplanar when these three vectors are coplanar (linearly dependent).

$$\begin{vmatrix} 2 & 2 & -2 \\ 3 & 0 & 3 \\ 1 & -2 & 5 \end{vmatrix} = 2(6) - 2(12) + (-2)(-6)$$

$= 0$

The determinant is zero, so $\overrightarrow{AB}, \overrightarrow{AC}, \overrightarrow{AD}$ are coplanar and A, B, C, D are coplanar.

Ans: The determinant is 0**, so** A, B, C, D **are coplanar**

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11. The midpoint of AB is $(2, -\frac{3}{2}, 1)$ and $A = (1,1,7)$. Find B. [1]

$\frac{\bar{a}+\bar{b}}{2} = \bar{m} \Rightarrow \bar{b} = 2\bar{m} - \bar{a}$

$= (4,-3,2) - (1,1,7) = (3,-4,-5)$

Ans: $(3,-4,-5)$

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12. Find the point that divides the segment joining $A(-3,-1,2)$ and $B(3,-5,3)$ externally in the ratio $3:2$. [2]

$\bar{r} = \frac{m\bar{b}-n\bar{a}}{m-n}$
$= \frac{3(3,-5,3)-2(-3,-1,2)}{3-2}$
$= \frac{(15,-13,5)}{1} = (15,-13,5)$
Ans: $(15,-13,5)$

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13. Find the points of trisection of the segment joining $A(5,0,-3)$ and $B(11,-6,-3)$. [3]

The points of trisection divide AB internally in the ratios $1:2$ and $2:1$.
$\frac{1(11,-6,-3)+2(5,0,-3)}{3} = \frac{(21,-6,-9)}{3} = (7,-2,-3)$
$\frac{2(11,-6,-3)+1(5,0,-3)}{3} = \frac{(27,-12,-9)}{3} = (9,-4,-3)$
Ans: $(7,-2,-3)$ **and** $(9,-4,-3)$

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14. Find the incentre of the triangle whose vertices are $A(-2,18,2)$, $B(-2,2,2)$ and $C(-2,10,8)$. [5]

$a = BC = \sqrt{0+64+36} = 10$
$b = CA = \sqrt{0+64+36} = 10$
$c = AB = \sqrt{0+256+0} = 16$
Incentre $I = \frac{a\bar{a}+b\bar{b}+c\bar{c}}{a+b+c}$
$= \frac{10(-2,18,2)+10(-2,2,2)+16(-2,10,8)}{36}$
$= \frac{(-72,360,168)}{36} = (-2,10,\frac{14}{3})$
Ans: $(-2,10,\frac{14}{3})$

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15. Find the vector and cartesian equations of the line through $A(4,2,0)$ and $B(1,5,3)$. [3]

$\bar{r} = (1-t)\bar{a} + t\bar{b} = (1-t)(4\bar{i}+2\bar{j}) + t(\bar{i}+5\bar{j}+3\bar{k})$
$= (4-3t)\bar{i} + (2+3t)\bar{j} + (3t)\bar{k}$
Direction $\overline{AB} = (-3,3,3)$
$\frac{x-4}{-3} = \frac{y-2}{3} = \frac{z}{3}$
Ans: $\bar{r} = (1-t)(4\bar{i}+2\bar{j}) + t(\bar{i}+5\bar{j}+3\bar{k})$; $\frac{x-4}{-3} = \frac{y-2}{3} = \frac{z}{3}$

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16. Write the vector equation of the plane through the points $A(4,1,1)$, $B(3,2,1)$ and $C(-3,-3,-2)$, and find its point with $s=1$, $t=2$. [2]

$\bar{r} = (1-s-t)\bar{a} + s\bar{b} + t\bar{c}$
$\bar{r} = (1-s-t)(4\bar{i}+\bar{j}+\bar{k}) + s(3\bar{i}+2\bar{j}+\bar{k}) + t(-3\bar{i}-3\bar{j}-2\bar{k})$
$s=1,\ t=2:\ \bar{r} = -2(4,1,1) + 1(3,2,1) + 2(-3,-3,-2) = (-11,-6,-5)$
Ans: $\bar{r} = (1-s-t)(4\bar{i}+\bar{j}+\bar{k}) + s(3\bar{i}+2\bar{j}+\bar{k}) + t(-3\bar{i}-3\bar{j}-2\bar{k})$; **point** $(-11,-6,-5)$

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17. Find a vector along the external bisector of the angle between $\bar{a} = 8\bar{i} - 15\bar{k}$ and $\bar{b} = 7\bar{i} + 4\bar{j} + 4\bar{k}$. [2]

$|\bar{a}| = \sqrt{64+0+225} = 17, \quad |\bar{b}| = \sqrt{49+16+16} = 9$
$\frac{\bar{a}}{|\bar{a}|} - \frac{\bar{b}}{|b|} = \frac{1}{17}(8\bar{i}-15\bar{k}) - \frac{1}{9}(7\bar{i}+4\bar{j}+4\bar{k})$
$= -\frac{47}{153}\bar{i} - \frac{4}{9}\bar{j} - \frac{203}{153}\bar{k}$
Any non-zero multiple of this vector is also along the external bisector.
Ans: $-\frac{47}{153}\bar{i} - \frac{4}{9}\bar{j} - \frac{203}{153}\bar{k}$ **(or any multiple)**

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18. In triangle ABC, S is the circumcentre and O the orthocentre, and $\overline{SA}+\overline{SB}+\overline{SC}=\overline{SO}$. Prove that $\overline{OA}+\overline{OB}+\overline{OC}=2\,\overline{OS}$. [5]

$\overline{OA}=\overline{SA}-\overline{SO}$, $\overline{OB}=\overline{SB}-\overline{SO}$, $\overline{OC}=\overline{SC}-\overline{SO}$

Adding:

$\overline{OA}+\overline{OB}+\overline{OC}=(\overline{SA}+\overline{SB}+\overline{SC})-3\,\overline{SO}$

$=\overline{SO}-3\,\overline{SO}=-2\,\overline{SO}$

$=2\,\overline{OS}$

Ans: Proved (see the working)

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19. Find the components of $\bar{a}=-\bar{i}+2\bar{j}+5\bar{k}$ when the coordinate axes are rotated about the x-axis through 60°. [3]

About the x-axis through α the new components are $(a_1,\ a_2\cos\alpha+a_3\sin\alpha,\ -a_2\sin\alpha+a_3\cos\alpha)$.

$\cos 60^\circ=\frac{1}{2},\quad \sin 60^\circ=\frac{\sqrt{3}}{2}$

Substituting $a_1=-1$, $a_2=2$, $a_3=5$:

$((-1),\ 2\cdot\frac{1}{2}+5\cdot\frac{\sqrt{3}}{2},\ -2\cdot\frac{\sqrt{3}}{2}+5\cdot\frac{1}{2})$

$=(-1,1+\frac{5\sqrt{3}}{2},\frac{5}{2}-\sqrt{3})$

Ans: $(-1,1+\frac{5\sqrt{3}}{2},\frac{5}{2}-\sqrt{3})$

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20. A straight line cuts the lines BC, CA, AB of triangle ABC at D, E, F. If $\frac{BD}{DC}=-\frac{5}{2}$ and $\frac{CE}{EA}=\frac{2}{3}$ (signed ratios), find $\frac{AF}{FB}$. [1]

By Menelaus' theorem $\frac{BD}{DC}\cdot\frac{CE}{EA}\cdot\frac{AF}{FB}=-1$

$\frac{AF}{FB}=\frac{-1}{(-\frac{5}{2})(\frac{2}{3})}=\frac{3}{5}$

Ans: $\frac{3}{5}$