

Trigonometry up to Transformations

Question Paper, Set 06

Time: 1 hour 30 minutes

Maximum marks: 50

Name: _____ Class: _____ Date: _____

1. Answer all 20 questions. The marks for each question are shown in brackets.
2. Show your working for the 2, 3 and 5 mark questions.
3. The key sheet and worked solutions are at the end of this paper.

1. If $\csc \theta = \frac{41}{9}$ and θ is acute, find $\cos \theta$ and $\tan \theta$. [2 marks]
2. Express 252° in radians. [1 mark]
3. Evaluate $\frac{1 - \tan^2 30^\circ}{1 + \tan^2 30^\circ}$. [2 marks]
4. Simplify $\frac{\sec(270^\circ + \theta) \cos(-\theta) \csc(90^\circ + \theta)}{\csc(360^\circ - \theta) \sin(90^\circ + \theta) \sec(180^\circ + \theta)}$. [3 marks]
5. Find the value of $\cot 690^\circ$. [1 mark]
6. Find the range of $4 - 3 \cos x$. [1 mark]
7. Which is greater: $\tan 1$ or $\tan 1^\circ$? [1 mark]
8. Find the value of $\cos 1^\circ + \cos 2^\circ + \cos 3^\circ + \dots + \cos 179^\circ$. [3 marks]
9. Simplify $\sqrt{1 - \cos^2 200^\circ}$. [2 marks]
10. If $\cos A = \frac{3}{5}$ and $\sin B = \frac{21}{29}$, where A and B are acute, find $\cos(A + B)$. [3 marks]
11. If $0 < A < B < \frac{\pi}{4}$, $\tan(A + B) = \frac{21}{20}$ and $\sin(A - B) = -\frac{5}{13}$, find $\tan 2B$. [5 marks]
12. Find the value of $\cos^2 45^\circ - \sin^2 15^\circ$. [1 mark]
13. If $A + B + C = 90^\circ$, $\tan A = \frac{1}{4}$ and $\tan B = \frac{1}{5}$, find $\tan C$. [2 marks]
14. If $\cos A = \frac{40}{41}$ and A is acute, find $\sin 2A$ and $\tan 2A$. [2 marks]
15. If $\tan \alpha = \frac{20}{21}$ and $\pi < \alpha < \frac{3\pi}{2}$, find $\sin 2\alpha$ and $\tan 3\alpha$. [5 marks]
16. Find the value of $\cos \frac{\pi}{13} + \cos \frac{3\pi}{13} + \cos \frac{5\pi}{13} + \dots + \cos \frac{11\pi}{13}$. [3 marks]
17. If $A - B = 60^\circ$, prove that $\cos^2 A + \cos^2 B - \cos A \cos B = \frac{3}{4}$. Hence find $\cos^2 80^\circ + \cos^2 20^\circ - \cos 80^\circ \cos 20^\circ$. [5 marks]
18. Simplify $\sqrt{1 + \sin 100^\circ}$. [1 mark]
19. In $\triangle ABC$, prove that $\sin^2 A + \sin^2 B + \sin^2 C = 2 + 2 \cos A \cos B \cos C$. [5 marks]

20. What is the period of $\cos^3 x$?

[2 marks]

Key sheet

Q	Answer
1	$\cos \theta = \frac{40}{41}, \tan \theta = \frac{9}{40}$
2	$\frac{7\pi}{5}$
3	$\frac{1}{2}$
4	1
5	$-\sqrt{3}$
6	$[1, 7]$
7	$\tan 1$ is greater
8	0
9	$-\sin 200^\circ$
10	$-\frac{24}{145}$
11	$\frac{352}{135}$
12	$\frac{\sqrt{3}}{4}$
13	$\frac{19}{9}$
14	$\sin 2A = \frac{720}{1681}, \tan 2A = \frac{720}{1519}$
15	$\sin 2\alpha = \frac{840}{841}, \tan 3\alpha = -\frac{18460}{15939}$
16	$\frac{1}{2}$
17	$\frac{3}{4}$
18	$\cos 50^\circ + \sin 50^\circ$
19	Proved: $\text{LHS} = 2 + 2 \cos A \cos B \cos C$
20	2π

Solutions

1. If $\csc \theta = \frac{41}{9}$ and θ is acute, find $\cos \theta$ and $\tan \theta$. [2]

$$\csc \theta = \frac{\text{hyp}}{\text{opp}} = \frac{41}{9}; \text{ take hyp} = 41, \text{ opp} = 9$$

$$\text{adj} = \sqrt{41^2 - 9^2} = \sqrt{1600} = 40$$

$$\cos \theta = \frac{\text{adj}}{\text{hyp}} = \frac{40}{41}, \tan \theta = \frac{\text{opp}}{\text{adj}} = \frac{9}{40}$$

$$\text{Ans: } \cos \theta = \frac{40}{41}, \tan \theta = \frac{9}{40}$$

2. Express 252° in radians. [1]

$$180^\circ = \pi, \text{ so } 1^\circ = \frac{\pi}{180}$$

$$252^\circ = 252 \times \frac{\pi}{180} = \frac{7\pi}{5}$$

$$\text{Ans: } \frac{7\pi}{5}$$

3. Evaluate $\frac{1 - \tan^2 30^\circ}{1 + \tan^2 30^\circ}$. [2]

$$\tan^2 30^\circ = \frac{1}{3}$$

$$= \frac{1 - \frac{1}{3}}{1 + \frac{1}{3}} = \frac{2/3}{4/3}$$

$$= \frac{1}{2}$$

$$\text{Ans: } \frac{1}{2}$$

4. Simplify $\frac{\sec(270^\circ + \theta) \cos(-\theta) \csc(90^\circ + \theta)}{\csc(360^\circ - \theta) \sin(90^\circ + \theta) \sec(180^\circ + \theta)}$. [3]

$$\text{Numerator: } \sec(270^\circ + \theta) = \csc \theta, \cos(-\theta) = \cos \theta, \csc(90^\circ + \theta) = \sec \theta$$

$$\text{Denominator: } \csc(360^\circ - \theta) = -\csc \theta, \sin(90^\circ + \theta) = \cos \theta, \sec(180^\circ + \theta) = -\sec \theta$$

$$\text{G.E} = \frac{(\csc \theta)(\cos \theta)(\sec \theta)}{(-\csc \theta)(\cos \theta)(-\sec \theta)} = 1$$

$$\text{Ans: } 1$$

5. Find the value of $\cot 690^\circ$. [1]

$$\cot 690^\circ = \cot(360^\circ + 330^\circ) = \cot 330^\circ$$

$$330^\circ \in \text{IV}_Q, \text{ where } \cot \text{ is -ve}$$

$$\cot 330^\circ = \cot(360^\circ - 30^\circ) = -\cot 30^\circ = -\sqrt{3}$$

$$\text{Ans: } -\sqrt{3}$$

6. Find the range of $4 - 3 \cos x$. [1]

$$-1 \leq \cos x \leq 1$$

$$\Rightarrow -3 \leq -3 \cos x \leq 3$$

$$\Rightarrow 1 \leq 4 - 3 \cos x \leq 7$$

$$\text{Range} = [1, 7]$$

$$\text{Ans: } [1, 7]$$

7. Which is greater: $\tan 1$ or $\tan 1^\circ$? [1]

$$1 = 1^\circ \approx 57^\circ$$

$$\text{From the graph, } \tan x \text{ increases from } 0^\circ \text{ to } 90^\circ$$

$$57^\circ > 1^\circ \Rightarrow \tan 1 > \tan 1^\circ$$

$$\text{Ans: } \tan 1 \text{ is greater}$$

8. Find the value of $\cos 1^\circ + \cos 2^\circ + \cos 3^\circ + \dots + \cos 179^\circ$. [3]

$$1^\circ + 179^\circ = 180^\circ: \cos 1^\circ + \cos 179^\circ = \cos 1^\circ - \cos 1^\circ = 0$$

The terms make 89 such pairs, and $\cos 90^\circ = 0$ is left

$$\text{G.E} = 0$$

Ans: 0

9. Simplify $\sqrt{1 - \cos^2 200^\circ}$. [2]

$$\sqrt{1 - \cos^2 200^\circ} = \sqrt{\sin^2 200^\circ} = |\sin 200^\circ|$$

$$200^\circ \in \text{III}_Q \Rightarrow \sin 200^\circ < 0$$

$$\therefore |\sin 200^\circ| = -\sin 200^\circ$$

Ans: $-\sin 200^\circ$

10. If $\cos A = \frac{3}{5}$ and $\sin B = \frac{21}{29}$, where A and B are acute, find $\cos(A + B)$. [3]

$$\cos A = \frac{3}{5} \Rightarrow \sin A = \frac{4}{5} \text{ (A acute)}$$

$$\sin B = \frac{21}{29} \Rightarrow \cos B = \frac{20}{29} \text{ (B acute)}$$

$$\cos(A + B) = \cos A \cos B - \sin A \sin B$$

$$= \frac{3}{5} \cdot \frac{20}{29} - \frac{4}{5} \cdot \frac{21}{29} = \frac{12}{29} - \frac{84}{145} = -\frac{24}{145}$$

Ans: $-\frac{24}{145}$

11. If $0 < A < B < \frac{\pi}{4}$, $\tan(A + B) = \frac{21}{20}$ and $\sin(A - B) = -\frac{5}{13}$, find $\tan 2B$. [5]

$$0 < A < B < \frac{\pi}{4} \Rightarrow 0 < A + B < \frac{\pi}{2} \text{ and } A - B < 0$$

$$\tan(A + B) = \frac{21}{20}$$

$$\sin(A - B) = -\frac{5}{13} \Rightarrow \cos(A - B) = \frac{12}{13}, \tan(A - B) = -\frac{5}{12}$$

$$\tan 2B = \tan[(A + B) - (A - B)] = \frac{\tan(A+B) - \tan(A-B)}{1 + \tan(A+B)\tan(A-B)}$$

$$= \frac{\frac{21}{20} - \left(-\frac{5}{12}\right)}{1 + \frac{21}{20} \cdot \left(-\frac{5}{12}\right)} = \frac{\frac{22}{16}}{1 - \frac{21 \cdot 5}{20 \cdot 12}}$$

$$= \frac{352}{135}$$

Ans: $\frac{352}{135}$

12. Find the value of $\cos^2 45^\circ - \sin^2 15^\circ$. [1]

$$\cos^2 A - \sin^2 B = \cos(A + B) \cos(A - B)$$

$$= \cos 60^\circ \cos 30^\circ$$

$$= \frac{1}{2} \cdot \frac{\sqrt{3}}{2}$$

$$= \frac{\sqrt{3}}{4}$$

Ans: $\frac{\sqrt{3}}{4}$

13. If $A + B + C = 90^\circ$, $\tan A = \frac{1}{4}$ and $\tan B = \frac{1}{5}$, find $\tan C$. [2]

$$\text{If } A + B + C = 90^\circ: \tan A \tan B + \tan B \tan C + \tan C \tan A = 1$$

$$\frac{1}{20} + \left(\frac{1}{4} + \frac{1}{5}\right) \tan C = 1$$

$$\frac{9}{20} \tan C = \frac{19}{20} \Rightarrow \tan C = \frac{19}{9}$$

Ans: $\frac{19}{9}$

14. If $\cos A = \frac{40}{41}$ and A is acute, find $\sin 2A$ and $\tan 2A$. [2]

$$\cos A = \frac{40}{41} \Rightarrow \sin A = \frac{9}{41}, \tan A = \frac{9}{40} \text{ (A acute)}$$

$$\sin 2A = 2 \sin A \cos A = 2 \cdot \frac{9}{41} \cdot \frac{40}{41} = \frac{720}{1681}$$

$$\tan 2A = \frac{2 \tan A}{1 - \tan^2 A} = \frac{\frac{18}{1600}}{1 - \frac{81}{1600}} = \frac{720}{1519}$$

$$\text{Ans: } \sin 2A = \frac{720}{1681}, \tan 2A = \frac{720}{1519}$$

15. If $\tan \alpha = \frac{20}{21}$ and $\pi < \alpha < \frac{3\pi}{2}$, find $\sin 2\alpha$ and $\tan 3\alpha$. [5]

$$\text{Given } \tan \alpha = \frac{20}{21} \text{ and } \alpha \in \text{III}_Q$$

$$\text{Right triangle with sides 20, 21, 29; with the signs of III}_Q: \sin \alpha = -\frac{20}{29}, \cos \alpha = -\frac{21}{29}, \tan \alpha = \frac{20}{21}$$

$$\sin 2\alpha = 2 \sin \alpha \cos \alpha = 2 \left(-\frac{20}{29}\right) \left(-\frac{21}{29}\right) = \frac{840}{841}$$

$$\tan 3\alpha = \frac{3 \tan \alpha - \tan^3 \alpha}{1 - 3 \tan^2 \alpha} = \frac{\frac{20}{7} - \frac{8000}{9261}}{1 - \frac{400}{147}} = -\frac{18460}{15939}$$

$$\text{Ans: } \sin 2\alpha = \frac{840}{841}, \tan 3\alpha = -\frac{18460}{15939}$$

16. Find the value of $\cos \frac{\pi}{13} + \cos \frac{3\pi}{13} + \cos \frac{5\pi}{13} + \dots + \cos \frac{11\pi}{13}$. [3]

$$\text{Angles in A.P.: } a_1 = \frac{\pi}{13}, a_n = \frac{11\pi}{13}, d = \frac{2\pi}{13}, n = 6$$

$$\text{Sum} = \frac{\cos \frac{a_1 + a_n}{2} \sin \frac{nd}{2}}{\sin \frac{d}{2}}$$

$$= \frac{\cos \frac{6\pi}{13} \sin \frac{6\pi}{13}}{\sin \frac{\pi}{13}} = \frac{\sin \frac{12\pi}{13}}{2 \sin \frac{\pi}{13}}$$

$$\sin \frac{12\pi}{13} = \sin \frac{\pi}{13}, \text{ so the sum} = \frac{1}{2}$$

$$\text{Ans: } \frac{1}{2}$$

17. If $A - B = 60^\circ$, prove that $\cos^2 A + \cos^2 B - \cos A \cos B = \frac{3}{4}$. Hence find $\cos^2 80^\circ + \cos^2 20^\circ - \cos 80^\circ \cos 20^\circ$. [5]

$$\cos^2 A + \cos^2 B = 1 + (\cos^2 A - \sin^2 B) = 1 + \cos(A + B) \cos(A - B) = 1 + \frac{1}{2} \cos(A + B)$$

$$\cos A \cos B = \frac{1}{2} [\cos(A + B) + \cos(A - B)] = \frac{1}{2} \cos(A + B) + \frac{1}{4}$$

$$\text{LHS} = 1 + \frac{1}{2} \cos(A + B) - \frac{1}{2} \cos(A + B) - \frac{1}{4} = \frac{3}{4} = \text{RHS}$$

$$80^\circ - 20^\circ = 60^\circ, \text{ so the value is } \frac{3}{4}$$

$$\text{Ans: } \frac{3}{4}$$

18. Simplify $\sqrt{1 + \sin 100^\circ}$. [1]

$$1 + \sin 100^\circ = \cos^2 50^\circ + \sin^2 50^\circ + 2 \sin 50^\circ \cos 50^\circ = (\cos 50^\circ + \sin 50^\circ)^2$$

$$\sqrt{1 + \sin 100^\circ} = |\cos 50^\circ + \sin 50^\circ| = \cos 50^\circ + \sin 50^\circ \text{ (both +ve)}$$

$$\text{Ans: } \cos 50^\circ + \sin 50^\circ$$

19. In $\triangle ABC$, prove that $\sin^2 A + \sin^2 B + \sin^2 C = 2 + 2 \cos A \cos B \cos C$. [5]

$$\text{Given } A + B + C = 180^\circ, \text{ so } \sin(A + B) = \sin C \text{ and } \cos(A + B) = -\cos C$$

$$\sin^2 A + \sin^2 B = 1 - (\cos^2 B - \sin^2 A) = 1 - \cos(A + B) \cos(A - B) = 1 + \cos C \cos(A - B)$$

$$\text{LHS} = 2 + \cos C \cos(A - B) - \cos^2 C = 2 + \cos C [\cos(A - B) + \cos(A + B)]$$

$$= 2 + \cos C \cdot 2 \cos A \cos B = 2 + 2 \cos A \cos B \cos C$$

$$= \text{RHS}$$

$$\text{Ans: Proved: LHS} = 2 + 2 \cos A \cos B \cos C$$

20. What is the period of $\cos^3 x$? [2]

$$\cos 3x = 4 \cos^3 x - 3 \cos x \Rightarrow \cos^3 x = \frac{3}{4} \cos x + \frac{1}{4} \cos 3x$$

$$\text{Period} = \text{L.C.M}\left\{2\pi, \frac{2\pi}{3}\right\} = \frac{\text{L.C.M}\{2\pi, 2\pi\}}{\text{G.C.D}\{1, 3\}} = 2\pi$$

Ans: 2π
