

Trigonometry up to Transformations

Question Paper, Set 05

Time: 1 hour 30 minutes

Maximum marks: 50

Name: _____ Class: _____ Date: _____

1. Answer all 20 questions. The marks for each question are shown in brackets.
2. Show your working for the 2, 3 and 5 mark questions.
3. The key sheet and worked solutions are at the end of this paper.

1. If $\cot \theta = \frac{21}{20}$ and θ is acute, find $\sin \theta$ and $\cos \theta$. [2 marks]
2. Express $\frac{3\pi}{5}$ radians in grades. [1 mark]
3. State whether $\sin 250^\circ \tan 250^\circ$ is positive or negative. [1 mark]
4. Evaluate $6 \cos^2 60^\circ - 4 \sin^2 45^\circ + 3 \tan^2 30^\circ$. [2 marks]
5. Simplify $\frac{\cos(90^\circ - \theta) \csc(360^\circ - \theta) \csc(90^\circ - \theta)}{\sin(180^\circ - \theta) \sec(90^\circ + \theta) \sec(360^\circ - \theta)}$. [3 marks]
6. Find the value of $\csc 1230^\circ$. [1 mark]
7. Find the greatest value of $2 \sin x - 9$. [1 mark]
8. Find the value of $\sin^2 6^\circ + \sin^2 12^\circ + \sin^2 18^\circ + \dots + \sin^2 84^\circ$. [3 marks]
9. If $\sin \theta \cos \theta = \frac{1}{4}$, find $\sin^2 \theta + \cos^4 \theta$. [2 marks]
10. If $\cos A = \frac{21}{29}$ and $\sin B = \frac{12}{13}$, where A and B are acute, find $\sin(A - B)$. [3 marks]
11. If $0 < A < B < \frac{\pi}{4}$, $\cos(A + B) = \frac{15}{17}$ and $\cos(A - B) = \frac{40}{41}$, find $\tan 2A$. [5 marks]
12. In $\triangle ABC$, $\tan A = 3$ and $\tan B = 4$. Find $\tan C$. [2 marks]
13. Find the value of $\tan 105^\circ + \tan 120^\circ + \tan 105^\circ \tan 120^\circ$. [1 mark]
14. If $\sin A = \frac{20}{29}$ and A is acute, find $\sin 2A$ and $\cos 2A$. [2 marks]
15. If $\cos \alpha = -\frac{15}{17}$ and $\frac{\pi}{2} < \alpha < \pi$, find $\sin 3\alpha$ and $\cos 2\alpha$. [5 marks]
16. Find the value of $4 \sin 18^\circ \cos 36^\circ$. [1 mark]
17. Find the value of $\tan 9^\circ + 2 \tan 18^\circ + 4 \cot 36^\circ$. [3 marks]
18. If $A + B = 60^\circ$, prove that $\sin^2 A + \sin^2 B + \sin A \sin B = \frac{3}{4}$. Hence find $\sin^2 25^\circ + \sin^2 35^\circ + \sin 25^\circ \sin 35^\circ$. [5 marks]
19. In $\triangle ABC$, prove that $\cos^2 A + \cos^2 B + \cos^2 C = 1 - 2 \cos A \cos B \cos C$. [5 marks]

20. Find the period of $5 \sec(3x + 2)$.

[2 marks]

Key sheet

Q	Answer
1	$\sin \theta = \frac{20}{29}, \cos \theta = \frac{21}{29}$
2	120°
3	Negative
4	$\frac{1}{2}$
5	1
6	2
7	-7
8	7
9	$\frac{15}{16}$
10	$-\frac{152}{377}$
11	$\frac{185}{672}$
12	$\frac{7}{11}$
13	1
14	$\sin 2A = \frac{840}{841}, \cos 2A = \frac{41}{841}$
15	$\sin 3\alpha = \frac{4888}{4913}, \cos 2\alpha = \frac{161}{289}$
16	1
17	$\cot 9^\circ$
18	$\frac{3}{4}$
19	Proved: $\text{LHS} = 1 - 2 \cos A \cos B \cos C$
20	$\frac{2\pi}{3}$

Solutions

1. If $\cot \theta = \frac{21}{20}$ and θ is acute, find $\sin \theta$ and $\cos \theta$. [2]

$$\cot \theta = \frac{\text{adj}}{\text{opp}} = \frac{21}{20}: \text{ take adj} = 21, \text{ opp} = 20$$

$$\text{hyp} = \sqrt{20^2 + 21^2} = \sqrt{841} = 29$$

$$\sin \theta = \frac{\text{opp}}{\text{hyp}} = \frac{20}{29}, \cos \theta = \frac{\text{adj}}{\text{hyp}} = \frac{21}{29}$$

$$\text{Ans: } \sin \theta = \frac{20}{29}, \cos \theta = \frac{21}{29}$$

2. Express $\frac{3\pi}{5}$ radians in grades. [1]

$$\pi = 200^g$$

$$\frac{3\pi}{5} = \frac{3}{5} \times 200^g = 120^g$$

$$\text{Ans: } 120^g$$

3. State whether $\sin 250^\circ \tan 250^\circ$ is positive or negative. [1]

$$250^\circ \in \text{III}_Q \Rightarrow \sin 250^\circ \text{ is -ve}$$

$$250^\circ \in \text{III}_Q \Rightarrow \tan 250^\circ \text{ is +ve}$$

So the product is negative

Ans: Negative

4. Evaluate $6 \cos^2 60^\circ - 4 \sin^2 45^\circ + 3 \tan^2 30^\circ$. [2]

$$= 6 \left(\frac{1}{2}\right)^2 - 4 \left(\frac{1}{\sqrt{2}}\right)^2 + 3 \left(\frac{1}{\sqrt{3}}\right)^2$$

$$= \frac{3}{2} - 2 + 1$$

$$= \frac{1}{2}$$

$$\text{Ans: } \frac{1}{2}$$

5. Simplify $\frac{\cos(90^\circ - \theta) \csc(360^\circ - \theta) \csc(90^\circ - \theta)}{\sin(180^\circ - \theta) \sec(90^\circ + \theta) \sec(360^\circ - \theta)}$. [3]

$$\text{Numerator: } \cos(90^\circ - \theta) = \sin \theta, \csc(360^\circ - \theta) = -\csc \theta, \csc(90^\circ - \theta) = \sec \theta$$

$$\text{Denominator: } \sin(180^\circ - \theta) = \sin \theta, \sec(90^\circ + \theta) = -\csc \theta, \sec(360^\circ - \theta) = \sec \theta$$

$$\text{G.E} = \frac{(\sin \theta)(-\csc \theta)(\sec \theta)}{(\sin \theta)(-\csc \theta)(\sec \theta)} = 1$$

$$\text{Ans: } 1$$

6. Find the value of $\csc 1230^\circ$. [1]

$$\csc 1230^\circ = \csc(1080^\circ + 150^\circ) = \csc 150^\circ$$

$$150^\circ \in \text{II}_Q, \text{ where } \csc \text{ is +ve}$$

$$\csc 150^\circ = \csc(180^\circ - 30^\circ) = +\csc 30^\circ = 2$$

$$\text{Ans: } 2$$

7. Find the greatest value of $2 \sin x - 9$. [1]

$$-1 \leq \sin x \leq 1$$

$$\Rightarrow -2 \leq 2 \sin x \leq 2$$

$$\Rightarrow -11 \leq 2 \sin x - 9 \leq -7$$

$$\text{Greatest value} = -7 \text{ (when } \sin x = 1)$$

$$\text{Ans: } -7$$

8. Find the value of $\sin^2 6^\circ + \sin^2 12^\circ + \sin^2 18^\circ + \dots + \sin^2 84^\circ$. [3]

$$\sin^2 6^\circ + \sin^2 84^\circ = \sin^2 6^\circ + \cos^2 6^\circ = 1$$

There are 14 terms (6° to 84° in steps of 6°): 7 pairs, each = 1

$$\text{G.E} = 7(1) = 7$$

Ans: 7

9. If $\sin \theta \cos \theta = \frac{1}{4}$, find $\sin^2 \theta + \cos^4 \theta$. [2]

$$\sin^2 \theta + \cos^4 \theta = \sin^2 \theta + \cos^2 \theta (1 - \sin^2 \theta) = 1 - \sin^2 \theta \cos^2 \theta$$

$$= 1 - \left(\frac{1}{4}\right)^2 = 1 - \frac{1}{16} = \frac{15}{16}$$

Ans: $\frac{15}{16}$

10. If $\cos A = \frac{21}{29}$ and $\sin B = \frac{12}{13}$, where A and B are acute, find $\sin(A - B)$. [3]

$$\cos A = \frac{21}{29} \Rightarrow \sin A = \frac{20}{29} \text{ (A acute)}$$

$$\sin B = \frac{12}{13} \Rightarrow \cos B = \frac{5}{13} \text{ (B acute)}$$

$$\sin(A - B) = \sin A \cos B - \cos A \sin B$$

$$= \frac{20}{29} \cdot \frac{5}{13} - \frac{21}{29} \cdot \frac{12}{13} = \frac{100}{377} - \frac{252}{377} = -\frac{152}{377}$$

Ans: $-\frac{152}{377}$

11. If $0 < A < B < \frac{\pi}{4}$, $\cos(A + B) = \frac{15}{17}$ and $\cos(A - B) = \frac{40}{41}$, find $\tan 2A$. [5]

$$0 < A < B < \frac{\pi}{4} \Rightarrow 0 < A + B < \frac{\pi}{2} \text{ and } A - B < 0$$

$$\cos(A + B) = \frac{15}{17} \Rightarrow \sin(A + B) = \frac{8}{17}, \tan(A + B) = \frac{8}{15}$$

$$\cos(A - B) = \frac{40}{41} \text{ and } A - B < 0 \Rightarrow \sin(A - B) = -\frac{9}{41}, \tan(A - B) = -\frac{9}{40}$$

$$\tan 2A = \tan[(A + B) + (A - B)] = \frac{\tan(A+B) + \tan(A-B)}{1 - \tan(A+B)\tan(A-B)}$$

$$= \frac{\frac{8}{15} - \frac{9}{40}}{1 + \frac{8}{15} \cdot \frac{9}{40}} = \frac{\frac{37}{120}}{\frac{25}{25}} = \frac{37}{120}$$

$$= \frac{185}{672}$$

Ans: $\frac{185}{672}$

12. In $\triangle ABC$, $\tan A = 3$ and $\tan B = 4$. Find $\tan C$. [2]

$$\text{In } \triangle ABC: \tan A + \tan B + \tan C = \tan A \tan B \tan C$$

$$3 + 4 + \tan C = 12 \tan C$$

$$(12 - 1) \tan C = 7 \Rightarrow \tan C = \frac{7}{11}$$

Ans: $\frac{7}{11}$

13. Find the value of $\tan 105^\circ + \tan 120^\circ + \tan 105^\circ \tan 120^\circ$. [1]

$$105^\circ + 120^\circ = 225^\circ, \tan 225^\circ = 1$$

$$\frac{\tan 105^\circ + \tan 120^\circ}{1 - \tan 105^\circ \tan 120^\circ} = 1$$

$$\Rightarrow \tan 105^\circ + \tan 120^\circ + \tan 105^\circ \tan 120^\circ = 1$$

Ans: 1

14. If $\sin A = \frac{20}{29}$ and A is acute, find $\sin 2A$ and $\cos 2A$. [2]

$$\sin A = \frac{20}{29} \Rightarrow \cos A = \frac{21}{29}, \tan A = \frac{20}{21} \text{ (} A \text{ acute)}$$

$$\sin 2A = 2 \sin A \cos A = 2 \cdot \frac{20}{29} \cdot \frac{21}{29} = \frac{840}{841}$$

$$\cos 2A = 1 - 2 \sin^2 A = 1 - 2 \cdot \frac{400}{841} = \frac{41}{841}$$

$$\text{Ans: } \sin 2A = \frac{840}{841}, \cos 2A = \frac{41}{841}$$

15. If $\cos \alpha = -\frac{15}{17}$ and $\frac{\pi}{2} < \alpha < \pi$, find $\sin 3\alpha$ and $\cos 2\alpha$. [5]

$$\text{Given } \cos \alpha = -\frac{15}{17} \text{ and } \alpha \in \Pi_Q$$

$$\text{Right triangle with sides 8, 15, 17; with the signs of } \Pi_Q: \sin \alpha = \frac{8}{17}, \cos \alpha = -\frac{15}{17}, \tan \alpha = -\frac{8}{15}$$

$$\sin 3\alpha = 3 \sin \alpha - 4 \sin^3 \alpha = 3 \left(\frac{8}{17} \right) - 4 \left(\frac{8}{17} \right)^3 = \frac{24}{17} - \frac{2048}{4913} = \frac{4888}{4913}$$

$$\cos 2\alpha = \cos^2 \alpha - \sin^2 \alpha = \frac{225}{289} - \frac{64}{289} = \frac{161}{289}$$

$$\text{Ans: } \sin 3\alpha = \frac{4888}{4913}, \cos 2\alpha = \frac{161}{289}$$

16. Find the value of $4 \sin 18^\circ \cos 36^\circ$. [1]

$$= 4 \cdot \frac{\sqrt{5}-1}{4} \cdot \frac{\sqrt{5}+1}{4}$$

$$= \frac{(\sqrt{5})^2 - 1^2}{4} = \frac{4}{4} = 1$$

$$\text{Ans: } 1$$

17. Find the value of $\tan 9^\circ + 2 \tan 18^\circ + 4 \cot 36^\circ$. [3]

$$\cot A - \tan A = 2 \cot 2A \Rightarrow \tan A = \cot A - 2 \cot 2A$$

$$\tan 9^\circ = \cot 9^\circ - 2 \cot 18^\circ, 2 \tan 18^\circ = 2 \cot 18^\circ - 4 \cot 36^\circ$$

$$\text{G.E} = \cot 9^\circ - 2 \cot 18^\circ + 2 \cot 18^\circ - 4 \cot 36^\circ + 4 \cot 36^\circ = \cot 9^\circ$$

$$\text{Ans: } \cot 9^\circ$$

18. If $A + B = 60^\circ$, prove that $\sin^2 A + \sin^2 B + \sin A \sin B = \frac{3}{4}$. Hence find $\sin^2 25^\circ + \sin^2 35^\circ + \sin 25^\circ \sin 35^\circ$. [5]

$$\sin^2 A + \sin^2 B = 1 - (\cos^2 A - \sin^2 B) = 1 - \cos(A+B) \cos(A-B) = 1 - \frac{1}{2} \cos(A-B)$$

$$\sin A \sin B = \frac{1}{2} [\cos(A-B) - \cos(A+B)] = \frac{1}{2} \cos(A-B) - \frac{1}{4}$$

$$\text{LHS} = 1 - \frac{1}{2} \cos(A-B) + \frac{1}{2} \cos(A-B) - \frac{1}{4} = \frac{3}{4} = \text{RHS}$$

$$25^\circ + 35^\circ = 60^\circ, \text{ so the value is } \frac{3}{4}$$

$$\text{Ans: } \frac{3}{4}$$

19. In $\triangle ABC$, prove that $\cos^2 A + \cos^2 B + \cos^2 C = 1 - 2 \cos A \cos B \cos C$. [5]

$$\text{Given } A + B + C = 180^\circ, \text{ so } \sin(A+B) = \sin C \text{ and } \cos(A+B) = -\cos C$$

$$\text{LHS} = 1 + (\cos^2 A - \sin^2 B) + \cos^2 C = 1 + \cos(A+B) \cos(A-B) + \cos^2 C$$

$$= 1 - \cos C [\cos(A-B) - \cos C] = 1 - \cos C [\cos(A-B) + \cos(A+B)]$$

$$= 1 - \cos C \cdot 2 \cos A \cos B = 1 - 2 \cos A \cos B \cos C$$

$$= \text{RHS}$$

$$\text{Ans: Proved: LHS} = 1 - 2 \cos A \cos B \cos C$$

20. Find the period of $5 \sec(3x + 2)$. [2]

Period of $\sec(ax + b)$ is $\frac{2\pi}{|a|}$

$$= \frac{2\pi}{3}$$

Ans: $\frac{2\pi}{3}$
