

Trigonometry up to Transformations

Question Paper, Set 04

Time: 1 hour 30 minutes

Maximum marks: 50

Name: _____ Class: _____ Date: _____

1. Answer all 20 questions. The marks for each question are shown in brackets.
2. Show your working for the 2, 3 and 5 mark questions.
3. The key sheet and worked solutions are at the end of this paper.

1. If $\cos \theta = \frac{24}{25}$ and θ is acute, find $\tan \theta$ and $\csc \theta$. [2 marks]
2. Express 120° in degrees. [1 mark]
3. Evaluate $\frac{\tan 45^\circ}{\csc 30^\circ} + \frac{\sec 60^\circ}{\cot 45^\circ}$. [2 marks]
4. Simplify $\frac{\csc(270^\circ - \theta) \tan(180^\circ - \theta) \sec(90^\circ + \theta)}{\sec(180^\circ - \theta) \cot(90^\circ - \theta) \csc(360^\circ - \theta)}$. [3 marks]
5. Find the value of $\sec 840^\circ$. [1 mark]
6. Find the range of $2 \sin x + 3$. [1 mark]
7. Which is greater: $\sin 2$ or $\sin 2^\circ$? [1 mark]
8. Find the value of $\tan 1^\circ \tan 2^\circ \tan 3^\circ \cdots \tan 89^\circ$. [3 marks]
9. Find the least value of $4 \sin^2 \theta + \csc^2 \theta$. [2 marks]
10. If $\sin A = \frac{12}{13}$ and $\cos B = \frac{8}{17}$, where A and B are acute, find $\cos(A - B)$. [3 marks]
11. If $0 < A < B < \frac{\pi}{4}$, $\sin(A + B) = \frac{3}{5}$ and $\sin(A - B) = -\frac{7}{25}$, find $\tan 2B$. [5 marks]
12. Find the value of $\sin^2 67\frac{1}{2}^\circ - \sin^2 22\frac{1}{2}^\circ$. [1 mark]
13. In $\triangle ABC$, $\cot A = 1$ and $\cot B = 3$. Find $\cot C$. [2 marks]
14. If $\tan A = \frac{7}{24}$ and A is acute, find $\cos 2A$ and $\tan 2A$. [2 marks]
15. If $\sin \alpha = -\frac{7}{25}$ and $\frac{3\pi}{2} < \alpha < 2\pi$, find $\cos 3\alpha$ and $\tan 2\alpha$. [5 marks]
16. Find the value of $\cos \frac{2\pi}{11} + \cos \frac{4\pi}{11} + \cos \frac{6\pi}{11} + \cos \frac{8\pi}{11} + \cos \frac{10\pi}{11}$. [3 marks]
17. Prove that $\sin^2 \theta + \sin^2(120^\circ + \theta) + \sin^2(120^\circ - \theta) = \frac{3}{2}$. Hence find $\sin^2 10^\circ + \sin^2 110^\circ + \sin^2 130^\circ$. [5 marks]
18. If $\frac{x}{y} = \frac{\tan 75^\circ}{\tan 15^\circ}$, find $\frac{x+y}{x-y}$. [1 mark]
19. In $\triangle ABC$, prove that $\sin A + \sin B - \sin C = 4 \sin \frac{A}{2} \sin \frac{B}{2} \cos \frac{C}{2}$. [5 marks]

20. Find the period of $\cos^2 x$.

[2 marks]

Key sheet

Q	Answer
1	$\tan \theta = \frac{7}{24}, \csc \theta = \frac{25}{7}$
2	108°
3	$\frac{5}{2}$
4	-1
5	-2
6	$[1, 5]$
7	$\sin 2$ is greater
8	1
9	4
10	$\frac{220}{221}$
11	$\frac{4}{3}$
12	$\frac{1}{\sqrt{2}}$
13	$-\frac{1}{2}$
14	$\cos 2A = \frac{527}{625}, \tan 2A = \frac{336}{527}$
15	$\cos 3\alpha = \frac{10296}{15625}, \tan 2\alpha = -\frac{336}{527}$
16	$-\frac{1}{2}$
17	$\frac{3}{2}$
18	$\frac{2}{\sqrt{3}}$
19	Proved: $\text{LHS} = 4 \sin \frac{A}{2} \sin \frac{B}{2} \cos \frac{C}{2}$
20	π

Solutions

1. If $\cos \theta = \frac{24}{25}$ and θ is acute, find $\tan \theta$ and $\csc \theta$. [2]

$$\cos \theta = \frac{\text{adj}}{\text{hyp}} = \frac{24}{25}; \text{ take adj} = 24, \text{ hyp} = 25$$

$$\text{opp} = \sqrt{25^2 - 24^2} = \sqrt{49} = 7$$

$$\tan \theta = \frac{\text{opp}}{\text{adj}} = \frac{7}{24}, \csc \theta = \frac{\text{hyp}}{\text{opp}} = \frac{25}{7}$$

$$\text{Ans: } \tan \theta = \frac{7}{24}, \csc \theta = \frac{25}{7}$$

2. Express 120^g in degrees. [1]

$$200^g = 180^\circ, \text{ so } 1^g = \frac{9}{10} \text{ degree}$$

$$120^g = 120 \times \frac{9}{10} = 108^\circ$$

$$\text{Ans: } 108^\circ$$

3. Evaluate $\frac{\tan 45^\circ}{\csc 30^\circ} + \frac{\sec 60^\circ}{\cot 45^\circ}$. [2]

$$= \frac{1}{2} + \frac{2}{1}$$

$$= \frac{5}{2}$$

$$\text{Ans: } \frac{5}{2}$$

4. Simplify $\frac{\csc(270^\circ - \theta) \tan(180^\circ - \theta) \sec(90^\circ + \theta)}{\sec(180^\circ - \theta) \cot(90^\circ - \theta) \csc(360^\circ - \theta)}$. [3]

$$\text{Numerator: } \csc(270^\circ - \theta) = -\sec \theta, \tan(180^\circ - \theta) = -\tan \theta, \sec(90^\circ + \theta) = -\csc \theta$$

$$\text{Denominator: } \sec(180^\circ - \theta) = -\sec \theta, \cot(90^\circ - \theta) = \tan \theta, \csc(360^\circ - \theta) = -\csc \theta$$

$$\text{G.E} = \frac{(-\sec \theta)(-\tan \theta)(-\csc \theta)}{(-\sec \theta)(\tan \theta)(-\csc \theta)} = -1$$

$$\text{Ans: } -1$$

5. Find the value of $\sec 840^\circ$. [1]

$$\sec 840^\circ = \sec(720^\circ + 120^\circ) = \sec 120^\circ$$

$$120^\circ \in \text{II}_Q, \text{ where sec is -ve}$$

$$\sec 120^\circ = \sec(180^\circ - 60^\circ) = -\sec 60^\circ = -2$$

$$\text{Ans: } -2$$

6. Find the range of $2 \sin x + 3$. [1]

$$-1 \leq \sin x \leq 1$$

$$\Rightarrow -2 \leq 2 \sin x \leq 2$$

$$\Rightarrow 1 \leq 2 \sin x + 3 \leq 5$$

$$\text{Range} = [1, 5]$$

$$\text{Ans: } [1, 5]$$

7. Which is greater: $\sin 2$ or $\sin 2^\circ$? [1]

$$2 = 2^c \approx 114^\circ$$

$$\sin 114^\circ = \sin(180^\circ - 66^\circ) = \sin 66^\circ$$

$$\sin x \text{ increases from } 0^\circ \text{ to } 90^\circ \text{ and } 2^\circ < 66^\circ, \text{ so } \sin 2^\circ < \sin 2$$

$$\text{Ans: } \sin 2 \text{ is greater}$$

8. Find the value of $\tan 1^\circ \tan 2^\circ \tan 3^\circ \cdots \tan 89^\circ$. [3]

$$\tan 1^\circ \tan 89^\circ = \tan 1^\circ \cot 1^\circ = 1 \text{ (as } \tan(90^\circ - \theta) = \cot \theta)$$

The 89 factors make 44 such pairs, and $\tan 45^\circ = 1$ is left

$$\text{G.E} = 1^{44} \times 1 = 1$$

Ans: 1

9. Find the least value of $4 \sin^2 \theta + \csc^2 \theta$. [2]

Consider $4 \sin^2 \theta$ and $\csc^2 \theta$; both are +ve

$$\text{A.M} \geq \text{G.M: } \frac{4 \sin^2 \theta + \csc^2 \theta}{2} \geq \sqrt{4 \sin^2 \theta \cdot \csc^2 \theta} = \sqrt{4} = 2$$

$$\Rightarrow 4 \sin^2 \theta + \csc^2 \theta \geq 4; \text{ equality when } \sin^2 \theta = \frac{1}{2}$$

Least value = 4

Ans: 4

10. If $\sin A = \frac{12}{13}$ and $\cos B = \frac{8}{17}$, where A and B are acute, find $\cos(A - B)$. [3]

$$\sin A = \frac{12}{13} \Rightarrow \cos A = \frac{5}{13} \text{ (A acute)}$$

$$\cos B = \frac{8}{17} \Rightarrow \sin B = \frac{15}{17} \text{ (B acute)}$$

$$\cos(A - B) = \cos A \cos B + \sin A \sin B$$

$$= \frac{5}{13} \cdot \frac{8}{17} + \frac{12}{13} \cdot \frac{15}{17} = \frac{40}{221} + \frac{180}{221} = \frac{220}{221}$$

Ans: $\frac{220}{221}$

11. If $0 < A < B < \frac{\pi}{4}$, $\sin(A + B) = \frac{3}{5}$ and $\sin(A - B) = -\frac{7}{25}$, find $\tan 2B$. [5]

$$0 < A < B < \frac{\pi}{4} \Rightarrow 0 < A + B < \frac{\pi}{2} \text{ and } A - B < 0$$

$$\sin(A + B) = \frac{3}{5} \Rightarrow \cos(A + B) = \frac{4}{5}, \tan(A + B) = \frac{3}{4}$$

$$\sin(A - B) = -\frac{7}{25} \Rightarrow \cos(A - B) = \frac{24}{25}, \tan(A - B) = -\frac{7}{24}$$

$$\tan 2B = \tan[(A + B) - (A - B)] = \frac{\tan(A+B) - \tan(A-B)}{1 + \tan(A+B) \tan(A-B)}$$

$$= \frac{\frac{3}{4} - \left(-\frac{7}{24}\right)}{1 + \frac{3}{4} \cdot \left(-\frac{7}{24}\right)} = \frac{\frac{25}{24}}{\frac{25}{32}}$$

$$= \frac{4}{3}$$

Ans: $\frac{4}{3}$

12. Find the value of $\sin^2 67\frac{1}{2}^\circ - \sin^2 22\frac{1}{2}^\circ$. [1]

$$\sin^2 A - \sin^2 B = \sin(A + B) \sin(A - B)$$

$$= \sin 90^\circ \sin 45^\circ$$

$$= 1 \cdot \frac{1}{\sqrt{2}}$$

$$= \frac{1}{\sqrt{2}}$$

Ans: $\frac{1}{\sqrt{2}}$

13. In $\triangle ABC$, $\cot A = 1$ and $\cot B = 3$. Find $\cot C$. [2]

$$\text{In } \triangle ABC: \cot A \cot B + \cot B \cot C + \cot C \cot A = 1$$

$$3 + (1 + 3) \cot C = 1$$

$$4 \cot C = 1 - 3 \Rightarrow \cot C = -\frac{1}{2}$$

Ans: $-\frac{1}{2}$

14. If $\tan A = \frac{7}{24}$ and A is acute, find $\cos 2A$ and $\tan 2A$. [2]

$$\tan A = \frac{7}{24}, \tan^2 A = \frac{49}{576}$$

$$\cos 2A = \frac{1 - \tan^2 A}{1 + \tan^2 A} = \frac{1 - \frac{49}{576}}{1 + \frac{49}{576}} = \frac{527}{625}$$

$$\tan 2A = \frac{2 \tan A}{1 - \tan^2 A} = \frac{\frac{14}{24}}{1 - \frac{49}{576}} = \frac{336}{527}$$

$$\text{Ans: } \cos 2A = \frac{527}{625}, \tan 2A = \frac{336}{527}$$

15. If $\sin \alpha = -\frac{7}{25}$ and $\frac{3\pi}{2} < \alpha < 2\pi$, find $\cos 3\alpha$ and $\tan 2\alpha$. [5]

$$\text{Given } \sin \alpha = -\frac{7}{25} \text{ and } \alpha \in \text{IV}_Q$$

$$\text{Right triangle with sides 7, 24, 25; with the signs of IV}_Q: \sin \alpha = -\frac{7}{25}, \cos \alpha = \frac{24}{25}, \tan \alpha = -\frac{7}{24}$$

$$\cos 3\alpha = 4 \cos^3 \alpha - 3 \cos \alpha = 4 \left(\frac{24}{25}\right)^3 - 3 \left(\frac{24}{25}\right) = \frac{55296}{15625} - \frac{72}{25} = \frac{10296}{15625}$$

$$\tan 2\alpha = \frac{2 \tan \alpha}{1 - \tan^2 \alpha} = \frac{-\frac{14}{25}}{1 - \frac{49}{625}} = \frac{-\frac{14}{25}}{\frac{576}{625}} = -\frac{336}{527}$$

$$\text{Ans: } \cos 3\alpha = \frac{10296}{15625}, \tan 2\alpha = -\frac{336}{527}$$

16. Find the value of $\cos \frac{2\pi}{11} + \cos \frac{4\pi}{11} + \cos \frac{6\pi}{11} + \cos \frac{8\pi}{11} + \cos \frac{10\pi}{11}$. [3]

$$\text{Angles in A.P.: } a_1 = \frac{2\pi}{11}, a_n = \frac{10\pi}{11}, d = \frac{2\pi}{11}, n = 5$$

$$\text{Sum} = \frac{\cos \frac{a_1 + a_n}{2} \sin \frac{nd}{2}}{\sin \frac{d}{2}}$$

$$= \frac{\cos \frac{6\pi}{11} \sin \frac{5\pi}{11}}{\sin \frac{\pi}{11}}, \text{ and } \cos \frac{6\pi}{11} = -\cos \frac{5\pi}{11}$$

$$= -\frac{\sin \frac{10\pi}{11}}{2 \sin \frac{\pi}{11}} = -\frac{\sin \frac{\pi}{11}}{2 \sin \frac{\pi}{11}} = -\frac{1}{2}$$

$$\text{Ans: } -\frac{1}{2}$$

17. Prove that $\sin^2 \theta + \sin^2 (120^\circ + \theta) + \sin^2 (120^\circ - \theta) = \frac{3}{2}$. Hence find $\sin^2 10^\circ + \sin^2 110^\circ + \sin^2 130^\circ$. [5]

$$\text{Use } \sin^2 A = \frac{1 - \cos 2A}{2}$$

$$\text{LHS} = \frac{3}{2} - \frac{1}{2} [\cos 2\theta + \cos(240^\circ + 2\theta) + \cos(240^\circ - 2\theta)]$$

$$= \frac{3}{2} - \frac{1}{2} [\cos 2\theta + 2 \cos 240^\circ \cos 2\theta], \text{ and } \cos 240^\circ = -\frac{1}{2}$$

$$= \frac{3}{2} - \frac{1}{2} [\cos 2\theta - \cos 2\theta] = \frac{3}{2} = \text{RHS}$$

$$\text{Put } \theta = 10^\circ: \sin^2 10^\circ + \sin^2 130^\circ + \sin^2 110^\circ = \frac{3}{2}$$

$$\text{Ans: } \frac{3}{2}$$

18. If $\frac{x}{y} = \frac{\tan 75^\circ}{\tan 15^\circ}$, find $\frac{x+y}{x-y}$. [1]

$$\frac{x}{y} = \frac{\tan A}{\tan B} \Rightarrow \frac{x+y}{x-y} = \frac{\tan A + \tan B}{\tan A - \tan B} = \frac{\sin(A+B)}{\sin(A-B)}$$

$$= \frac{\sin 90^\circ}{\sin 60^\circ} = \frac{2}{\sqrt{3}}$$

$$\text{Ans: } \frac{2}{\sqrt{3}}$$

19. In $\triangle ABC$, prove that $\sin A + \sin B - \sin C = 4 \sin \frac{A}{2} \sin \frac{B}{2} \cos \frac{C}{2}$. [5]

$$\frac{A+B}{2} = 90^\circ - \frac{C}{2}, \text{ so } \sin \frac{A+B}{2} = \cos \frac{C}{2} \text{ and } \cos \frac{A+B}{2} = \sin \frac{C}{2}$$

$$\text{LHS} = 2 \sin \frac{A+B}{2} \cos \frac{A-B}{2} - 2 \sin \frac{C}{2} \cos \frac{C}{2} = 2 \cos \frac{C}{2} [\cos \frac{A-B}{2} - \sin \frac{C}{2}]$$

$$= 2 \cos \frac{C}{2} [\cos \frac{A-B}{2} - \cos \frac{A+B}{2}] = 2 \cos \frac{C}{2} \cdot 2 \sin \frac{A}{2} \sin \frac{B}{2}$$

$$= \text{RHS}$$

$$\text{Ans: Proved: LHS} = 4 \sin \frac{A}{2} \sin \frac{B}{2} \cos \frac{C}{2}$$

20. Find the period of $\cos^2 x$. [2]

$$\cos^2 x = \frac{1+\cos 2x}{2} = \frac{1}{2} + \frac{1}{2} \cos 2x$$

$$\text{Period} = \frac{2\pi}{2} = \pi$$

Ans: π
