

Trigonometry up to Transformations

Question Paper, Set 03

Time: 1 hour 30 minutes

Maximum marks: 50

Name: _____ Class: _____ Date: _____

1. Answer all 20 questions. The marks for each question are shown in brackets.
2. Show your working for the 2, 3 and 5 mark questions.
3. The key sheet and worked solutions are at the end of this paper.

1. If $\sec \theta = \frac{17}{15}$ and θ is acute, find $\sin \theta$ and $\cot \theta$. [2 marks]
2. Express 135° in grades. [1 mark]
3. State whether $\cot 160^\circ$ is positive or negative. [1 mark]
4. Evaluate $2 \csc^2 30^\circ - 3 \cot^2 45^\circ + \cos 0^\circ$. [2 marks]
5. Simplify $\frac{\sec(270^\circ - \theta) \cos(90^\circ + \theta) \cos(180^\circ + \theta)}{\csc(-\theta) \sin(360^\circ - \theta) \sin(90^\circ + \theta)}$. [3 marks]
6. Find the value of $\tan 1215^\circ$. [1 mark]
7. Find the least value of $4 + 5 \sin x$. [1 mark]
8. Find the value of $\cos^2 10^\circ + \cos^2 20^\circ + \cos^2 30^\circ + \dots + \cos^2 80^\circ$. [3 marks]
9. If $\sin \theta \cos \theta = \frac{2}{5}$, find $\sin^6 \theta + \cos^6 \theta$. [2 marks]
10. If $\tan A = \frac{7}{24}$ and $\tan B = \frac{4}{3}$, where A and B are acute, find $\sin(A - B)$. [3 marks]
11. If $0 < A < B < \frac{\pi}{4}$, $\tan(A + B) = \frac{20}{21}$ and $\cos(A - B) = \frac{15}{17}$, find $\tan 2A$. [5 marks]
12. In $\triangle ABC$, $\tan A = 1$ and $\tan B = 2$. Find $\tan C$. [2 marks]
13. Find the value of $(1 - \cot 26^\circ)(1 - \cot 19^\circ)$. [1 mark]
14. If $\cos A = \frac{15}{17}$ and A is acute, find $\cos 2A$ and $\sin 2A$. [2 marks]
15. If $\tan \alpha = -\frac{8}{15}$ and $\frac{3\pi}{2} < \alpha < 2\pi$, find $\tan 2\alpha$ and $\cos 2\alpha$. [5 marks]
16. Find the value of $\tan 67\frac{1}{2}^\circ$. [1 mark]
17. Find the value of $\cos 12^\circ \cos 24^\circ \cos 48^\circ \cos 96^\circ$. [3 marks]
18. Prove that $\cos^2 \theta + \cos^2(60^\circ + \theta) + \cos^2(60^\circ - \theta) = \frac{3}{2}$. Hence find $\cos^2 20^\circ + \cos^2 40^\circ + \cos^2 80^\circ$. [5 marks]
19. In $\triangle ABC$, prove that $\sin A + \sin B + \sin C = 4 \cos \frac{A}{2} \cos \frac{B}{2} \cos \frac{C}{2}$. [5 marks]

20. Find the period of $\cos \frac{2x}{3} + \sin \frac{x}{2}$.

[2 marks]

Key sheet

Q	Answer
1	$\sin \theta = \frac{8}{17}, \cot \theta = \frac{15}{8}$
2	150°
3	Negative
4	6
5	-1
6	-1
7	-1
8	4
9	$\frac{13}{25}$
10	$-\frac{3}{5}$
11	$\frac{132}{475}$
12	3
13	2
14	$\cos 2A = \frac{161}{289}, \sin 2A = \frac{240}{289}$
15	$\tan 2\alpha = -\frac{240}{161}, \cos 2\alpha = \frac{161}{289}$
16	$\sqrt{2} + 1$
17	$-\frac{1}{16}$
18	$\frac{3}{2}$
19	Proved: $\text{LHS} = 4 \cos \frac{A}{2} \cos \frac{B}{2} \cos \frac{C}{2}$
20	12π

Solutions

1. If $\sec \theta = \frac{17}{15}$ and θ is acute, find $\sin \theta$ and $\cot \theta$. [2]

$$\sec \theta = \frac{\text{hyp}}{\text{adj}} = \frac{17}{15}; \text{ take hyp} = 17, \text{ adj} = 15$$

$$\text{opp} = \sqrt{17^2 - 15^2} = \sqrt{64} = 8$$

$$\sin \theta = \frac{\text{opp}}{\text{hyp}} = \frac{8}{17}, \cot \theta = \frac{\text{adj}}{\text{opp}} = \frac{15}{8}$$

$$\text{Ans: } \sin \theta = \frac{8}{17}, \cot \theta = \frac{15}{8}$$

2. Express 135° in grades. [1]

$$180^\circ = 200^g, \text{ so } 1^\circ = \frac{10^g}{9}$$

$$135^\circ = 135 \times \frac{10}{9} = 150^g$$

$$\text{Ans: } 150^g$$

3. State whether $\cot 160^\circ$ is positive or negative. [1]

$$160^\circ \in \text{II}_Q \Rightarrow \cot 160^\circ \text{ is -ve}$$

$$\text{Ans: Negative}$$

4. Evaluate $2 \csc^2 30^\circ - 3 \cot^2 45^\circ + \cos 0^\circ$. [2]

$$= 2(2)^2 - 3(1)^2 + 1$$

$$= 8 - 3 + 1$$

$$= 6$$

$$\text{Ans: } 6$$

5. Simplify $\frac{\sec(270^\circ - \theta) \cos(90^\circ + \theta) \cos(180^\circ + \theta)}{\csc(-\theta) \sin(360^\circ - \theta) \sin(90^\circ + \theta)}$. [3]

$$\text{Numerator: } \sec(270^\circ - \theta) = -\csc \theta, \cos(90^\circ + \theta) = -\sin \theta, \cos(180^\circ + \theta) = -\cos \theta$$

$$\text{Denominator: } \csc(-\theta) = -\csc \theta, \sin(360^\circ - \theta) = -\sin \theta, \sin(90^\circ + \theta) = \cos \theta$$

$$\text{G.E} = \frac{(-\csc \theta)(-\sin \theta)(-\cos \theta)}{(-\csc \theta)(-\sin \theta)(\cos \theta)} = -1$$

$$\text{Ans: } -1$$

6. Find the value of $\tan 1215^\circ$. [1]

$$\tan 1215^\circ = \tan(1080^\circ + 135^\circ) = \tan 135^\circ$$

$$135^\circ \in \text{II}_Q, \text{ where } \tan \text{ is -ve}$$

$$\tan 135^\circ = \tan(180^\circ - 45^\circ) = -\tan 45^\circ = -1$$

$$\text{Ans: } -1$$

7. Find the least value of $4 + 5 \sin x$. [1]

$$-1 \leq \sin x \leq 1$$

$$\Rightarrow -5 \leq 5 \sin x \leq 5$$

$$\Rightarrow -1 \leq 4 + 5 \sin x \leq 9$$

$$\text{Least value} = -1 \text{ (when } \sin x = -1)$$

$$\text{Ans: } -1$$

8. Find the value of $\cos^2 10^\circ + \cos^2 20^\circ + \cos^2 30^\circ + \dots + \cos^2 80^\circ$. [3]

$$\cos^2 10^\circ + \cos^2 80^\circ = \cos^2 10^\circ + \sin^2 10^\circ = 1$$

The 8 terms make 4 pairs: $(10^\circ, 80^\circ), (20^\circ, 70^\circ), (30^\circ, 60^\circ), (40^\circ, 50^\circ)$

$$\text{G.E} = 4(1) = 4$$

Ans: 4

9. If $\sin \theta \cos \theta = \frac{2}{5}$, find $\sin^6 \theta + \cos^6 \theta$. [2]

$$\begin{aligned}\sin^6 \theta + \cos^6 \theta &= (\sin^2 \theta + \cos^2 \theta)^3 - 3 \sin^2 \theta \cos^2 \theta (\sin^2 \theta + \cos^2 \theta) = 1 - 3(\sin \theta \cos \theta)^2 \\ &= 1 - 3 \left(\frac{2}{5}\right)^2 = 1 - \frac{12}{25} = \frac{13}{25}\end{aligned}$$

Ans: $\frac{13}{25}$

10. If $\tan A = \frac{7}{24}$ and $\tan B = \frac{4}{3}$, where A and B are acute, find $\sin(A - B)$. [3]

$$\tan A = \frac{7}{24} \Rightarrow \sin A = \frac{7}{25}, \cos A = \frac{24}{25}$$

$$\tan B = \frac{4}{3} \Rightarrow \sin B = \frac{4}{5}, \cos B = \frac{3}{5}$$

$$\sin(A - B) = \sin A \cos B - \cos A \sin B$$

$$= \frac{7}{25} \cdot \frac{3}{5} - \frac{24}{25} \cdot \frac{4}{5} = \frac{21}{125} - \frac{96}{125} = -\frac{3}{5}$$

Ans: $-\frac{3}{5}$

11. If $0 < A < B < \frac{\pi}{4}$, $\tan(A + B) = \frac{20}{21}$ and $\cos(A - B) = \frac{15}{17}$, find $\tan 2A$. [5]

$$0 < A < B < \frac{\pi}{4} \Rightarrow 0 < A + B < \frac{\pi}{2} \text{ and } A - B < 0$$

$$\tan(A + B) = \frac{20}{21}$$

$$\cos(A - B) = \frac{15}{17} \text{ and } A - B < 0 \Rightarrow \sin(A - B) = -\frac{8}{17}, \tan(A - B) = -\frac{8}{15}$$

$$\tan 2A = \tan[(A + B) + (A - B)] = \frac{\tan(A+B) + \tan(A-B)}{1 - \tan(A+B)\tan(A-B)}$$

$$= \frac{\frac{20}{21} - \frac{8}{15}}{1 + \frac{20}{21} \cdot \frac{8}{15}} = \frac{\frac{44}{105}}{\frac{96}{63}}$$

$$= \frac{132}{475}$$

Ans: $\frac{132}{475}$

12. In $\triangle ABC$, $\tan A = 1$ and $\tan B = 2$. Find $\tan C$. [2]

$$\text{In } \triangle ABC: \tan A + \tan B + \tan C = \tan A \tan B \tan C$$

$$1 + 2 + \tan C = 2 \tan C$$

$$(2 - 1) \tan C = 3 \Rightarrow \tan C = 3$$

Ans: 3

13. Find the value of $(1 - \cot 26^\circ)(1 - \cot 19^\circ)$. [1]

$$26^\circ + 19^\circ = 45^\circ$$

If $A + B = 45^\circ$ then $(1 - \cot A)(1 - \cot B) = 2$ (replace $\tan$ by $\cot$ in the result)

Ans: 2

14. If $\cos A = \frac{15}{17}$ and A is acute, find $\cos 2A$ and $\sin 2A$. [2]

$$\cos A = \frac{15}{17} \Rightarrow \sin A = \frac{8}{17}, \tan A = \frac{8}{15} \text{ (A acute)}$$

$$\cos 2A = 2 \cos^2 A - 1 = 2 \cdot \frac{225}{289} - 1 = \frac{161}{289}$$

$$\sin 2A = 2 \sin A \cos A = 2 \cdot \frac{8}{17} \cdot \frac{15}{17} = \frac{240}{289}$$

Ans: $\cos 2A = \frac{161}{289}, \sin 2A = \frac{240}{289}$

15. If $\tan \alpha = -\frac{8}{15}$ and $\frac{3\pi}{2} < \alpha < 2\pi$, find $\tan 2\alpha$ and $\cos 2\alpha$. [5]

Given $\tan \alpha = -\frac{8}{15}$ and $\alpha \in IV_Q$

Right triangle with sides 8, 15, 17; with the signs of IV_Q : $\sin \alpha = -\frac{8}{17}$, $\cos \alpha = \frac{15}{17}$, $\tan \alpha = -\frac{8}{15}$

$$\tan 2\alpha = \frac{2 \tan \alpha}{1 - \tan^2 \alpha} = \frac{-\frac{16}{15}}{1 - \frac{64}{225}} = \frac{-\frac{16}{15}}{\frac{161}{225}} = -\frac{240}{161}$$

$$\cos 2\alpha = \cos^2 \alpha - \sin^2 \alpha = \frac{225}{289} - \frac{64}{289} = \frac{161}{289}$$

$$\text{Ans: } \tan 2\alpha = -\frac{240}{161}, \cos 2\alpha = \frac{161}{289}$$

16. Find the value of $\tan 67\frac{1}{2}^\circ$. [1]

$$\tan^2 \frac{A}{2} = \frac{1 - \cos A}{1 + \cos A} \text{ with } A = 135^\circ, \cos 135^\circ = -\frac{1}{\sqrt{2}}$$

$$\tan^2 67\frac{1}{2}^\circ = \frac{1 + \frac{1}{\sqrt{2}}}{1 - \frac{1}{\sqrt{2}}} = \frac{\sqrt{2} + 1}{\sqrt{2} - 1} = (\sqrt{2} + 1)^2$$

$$\tan 67\frac{1}{2}^\circ = \sqrt{2} + 1 \text{ (positive, as } 67\frac{1}{2}^\circ \text{ is acute)}$$

$$\text{Ans: } \sqrt{2} + 1$$

17. Find the value of $\cos 12^\circ \cos 24^\circ \cos 48^\circ \cos 96^\circ$. [3]

$$\cos \theta \cos 2\theta \cos 4\theta \cos 8\theta = \frac{\sin 16\theta}{16 \sin \theta} \text{ with } \theta = 12^\circ$$

$$\text{G.E} = \frac{\sin 192^\circ}{16 \sin 12^\circ}$$

$$\sin 192^\circ = \sin(180^\circ + 12^\circ) = -\sin 12^\circ$$

$$\text{G.E} = -\frac{1}{16}$$

$$\text{Ans: } -\frac{1}{16}$$

18. Prove that $\cos^2 \theta + \cos^2(60^\circ + \theta) + \cos^2(60^\circ - \theta) = \frac{3}{2}$. Hence find $\cos^2 20^\circ + \cos^2 40^\circ + \cos^2 80^\circ$. [5]

$$\text{Use } \cos^2 A = \frac{1 + \cos 2A}{2}$$

$$\text{LHS} = \frac{3}{2} + \frac{1}{2}[\cos 2\theta + \cos(120^\circ + 2\theta) + \cos(120^\circ - 2\theta)]$$

$$= \frac{3}{2} + \frac{1}{2}[\cos 2\theta + 2 \cos 120^\circ \cos 2\theta] = \frac{3}{2} + \frac{1}{2}[\cos 2\theta - \cos 2\theta] = \frac{3}{2} = \text{RHS}$$

$$\text{Put } \theta = 20^\circ: \cos^2 20^\circ + \cos^2 80^\circ + \cos^2 40^\circ = \frac{3}{2}$$

$$\text{Ans: } \frac{3}{2}$$

19. In $\triangle ABC$, prove that $\sin A + \sin B + \sin C = 4 \cos \frac{A}{2} \cos \frac{B}{2} \cos \frac{C}{2}$. [5]

$$\frac{A+B}{2} = 90^\circ - \frac{C}{2}, \text{ so } \sin \frac{A+B}{2} = \cos \frac{C}{2} \text{ and } \cos \frac{A+B}{2} = \sin \frac{C}{2}$$

$$\text{LHS} = 2 \sin \frac{A+B}{2} \cos \frac{A-B}{2} + 2 \sin \frac{C}{2} \cos \frac{C}{2} = 2 \cos \frac{C}{2} [\cos \frac{A-B}{2} + \sin \frac{C}{2}]$$

$$= 2 \cos \frac{C}{2} [\cos \frac{A-B}{2} + \cos \frac{A+B}{2}] = 2 \cos \frac{C}{2} \cdot 2 \cos \frac{A}{2} \cos \frac{B}{2}$$

$$= \text{RHS}$$

$$\text{Ans: Proved: LHS} = 4 \cos \frac{A}{2} \cos \frac{B}{2} \cos \frac{C}{2}$$

20. Find the period of $\cos \frac{2x}{3} + \sin \frac{x}{2}$. [2]

$$\text{Period of } \cos \frac{2x}{3} = \frac{2\pi}{2/3} = 3\pi, \text{ period of } \sin \frac{x}{2} = 4\pi$$

$$\text{Period} = \text{L.C.M}\{3\pi, 4\pi\} = 12\pi$$

$$\text{Ans: } 12\pi$$