

Trigonometry up to Transformations

Question Paper, Set 02

Time: 1 hour 30 minutes

Maximum marks: 50

Name: _____ Class: _____ Date: _____

1. Answer all 20 questions. The marks for each question are shown in brackets.
2. Show your working for the 2, 3 and 5 mark questions.
3. The key sheet and worked solutions are at the end of this paper.

1. If $\tan \theta = \frac{5}{12}$ and θ is acute, find $\sin \theta$ and $\sec \theta$. [2 marks]
2. Express $\frac{7\pi}{9}$ radians in degrees. [1 mark]
3. Evaluate $\cos^2 30^\circ + \sin^2 45^\circ + \cot^2 60^\circ$. [2 marks]
4. Simplify $\frac{\csc(270^\circ + \theta) \tan(270^\circ - \theta) \sec(270^\circ - \theta)}{\sec(360^\circ - \theta) \cot(-\theta) \csc(-\theta)}$. [3 marks]
5. Find the value of $\cos 1500^\circ$. [1 mark]
6. Find the range of $5 \cos x$. [1 mark]
7. Which is greater: $\cos 1$ or $\cos 1^\circ$? [1 mark]
8. $ABCD$ is a cyclic quadrilateral. Find the value of $\tan A + \tan B + \tan C + \tan D$. [3 marks]
9. Find the least value of $9 \tan^2 \theta + 4 \cot^2 \theta$. [2 marks]
10. If $\cos A = \frac{15}{17}$ and $\sin B = \frac{20}{29}$, where A and B are acute, find $\cos(A + B)$. [3 marks]
11. If $0 < A < B < \frac{\pi}{4}$, $\cos(A + B) = \frac{4}{5}$ and $\sin(A - B) = -\frac{5}{13}$, find $\tan 2B$. [5 marks]
12. Find the value of $\sin^2 75^\circ - \sin^2 15^\circ$. [1 mark]
13. If $A + B + C = 90^\circ$, $\tan A = \frac{1}{2}$ and $\tan B = \frac{1}{3}$, find $\tan C$. [2 marks]
14. If $\sin A = \frac{5}{13}$ and A is acute, find $\sin 2A$ and $\tan 2A$. [2 marks]
15. If $\cos \alpha = -\frac{3}{5}$ and $\pi < \alpha < \frac{3\pi}{2}$, find $\cos 3\alpha$ and $\sin 2\alpha$. [5 marks]
16. Find the value of $\cos \frac{\pi}{9} + \cos \frac{3\pi}{9} + \cos \frac{5\pi}{9} + \cos \frac{7\pi}{9}$. [3 marks]
17. Prove that $\tan \theta \tan(60^\circ - \theta) \tan(60^\circ + \theta) = \tan 3\theta$. Hence find the value of $\tan 20^\circ \tan 40^\circ \tan 80^\circ$. [5 marks]
18. If $\frac{x}{y} = \frac{\sin 105^\circ}{\sin 15^\circ}$, find $\frac{x+y}{x-y}$. [1 mark]
19. In $\triangle ABC$, prove that $\cos 2A + \cos 2B + \cos 2C = -1 - 4 \cos A \cos B \cos C$. [5 marks]

20. Find the period of $3 \tan(4x - 1) + 2$.

[2 marks]

Key sheet

Q	Answer
1	$\sin \theta = \frac{5}{13}, \sec \theta = \frac{13}{12}$
2	140°
3	$\frac{19}{12}$
4	1
5	$\frac{1}{2}$
6	$[-5, 5]$
7	$\cos 1^\circ$ is greater
8	0
9	12
10	$\frac{155}{493}$
11	$\frac{56}{33}$
12	$\frac{\sqrt{3}}{2}$
13	1
14	$\sin 2A = \frac{120}{169}, \tan 2A = \frac{120}{119}$
15	$\cos 3\alpha = \frac{117}{125}, \sin 2\alpha = \frac{24}{25}$
16	$\frac{1}{2}$
17	$\sqrt{3}$
18	$\sqrt{3}$
19	Proved: $\text{LHS} = -1 - 4 \cos A \cos B \cos C$
20	$\frac{\pi}{4}$

Solutions

1. If $\tan \theta = \frac{5}{12}$ and θ is acute, find $\sin \theta$ and $\sec \theta$. [2]

$$\tan \theta = \frac{\text{opp}}{\text{adj}} = \frac{5}{12}: \text{take opp} = 5, \text{adj} = 12$$

$$\text{hyp} = \sqrt{5^2 + 12^2} = \sqrt{169} = 13$$

$$\sin \theta = \frac{\text{opp}}{\text{hyp}} = \frac{5}{13}, \sec \theta = \frac{\text{hyp}}{\text{adj}} = \frac{13}{12}$$

$$\text{Ans: } \sin \theta = \frac{5}{13}, \sec \theta = \frac{13}{12}$$

2. Express $\frac{7\pi}{9}$ radians in degrees. [1]

$$\pi = 180^\circ$$

$$\frac{7\pi}{9} = \frac{7}{9} \times 180^\circ = 140^\circ$$

$$\text{Ans: } 140^\circ$$

3. Evaluate $\cos^2 30^\circ + \sin^2 45^\circ + \cot^2 60^\circ$. [2]

$$= \left(\frac{\sqrt{3}}{2}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2 + \left(\frac{1}{\sqrt{3}}\right)^2$$

$$= \frac{3}{4} + \frac{1}{2} + \frac{1}{3} = \frac{9+6+4}{12}$$

$$= \frac{19}{12}$$

$$\text{Ans: } \frac{19}{12}$$

4. Simplify $\frac{\csc(270^\circ + \theta) \tan(270^\circ - \theta) \sec(270^\circ - \theta)}{\sec(360^\circ - \theta) \cot(-\theta) \csc(-\theta)}$. [3]

$$\text{Numerator: } \csc(270^\circ + \theta) = -\sec \theta, \tan(270^\circ - \theta) = \cot \theta, \sec(270^\circ - \theta) = -\csc \theta$$

$$\text{Denominator: } \sec(360^\circ - \theta) = \sec \theta, \cot(-\theta) = -\cot \theta, \csc(-\theta) = -\csc \theta$$

$$\text{G.E} = \frac{(-\sec \theta)(\cot \theta)(-\csc \theta)}{(\sec \theta)(-\cot \theta)(-\csc \theta)} = 1$$

$$\text{Ans: } 1$$

5. Find the value of $\cos 1500^\circ$. [1]

$$\cos 1500^\circ = \cos(1440^\circ + 60^\circ) = \cos 60^\circ$$

$$\cos 60^\circ = \frac{1}{2}$$

$$\text{Ans: } \frac{1}{2}$$

6. Find the range of $5 \cos x$. [1]

$$-1 \leq \cos x \leq 1$$

$$\Rightarrow -5 \leq 5 \cos x \leq 5$$

$$\text{Range} = [-5, 5]$$

$$\text{Ans: } [-5, 5]$$

7. Which is greater: $\cos 1$ or $\cos 1^\circ$? [1]

$$1 = 1^\circ \approx 57^\circ$$

From the graph, $\cos x$ decreases from 1 to 0 as x goes from 0° to 90°

$$1^\circ < 57^\circ \Rightarrow \cos 1^\circ > \cos 1$$

$$\text{Ans: } \cos 1^\circ \text{ is greater}$$

8. $ABCD$ is a cyclic quadrilateral. Find the value of $\tan A + \tan B + \tan C + \tan D$. [3]

In a cyclic quadrilateral $A + C = 180^\circ$ and $B + D = 180^\circ$

$$\tan C = \tan(180^\circ - A) = -\tan A, \tan D = \tan(180^\circ - B) = -\tan B$$

$$\text{G.E} = \tan A + \tan B - \tan A - \tan B = 0$$

Ans: 0

9. Find the least value of $9 \tan^2 \theta + 4 \cot^2 \theta$. [2]

Consider $9 \tan^2 \theta$ and $4 \cot^2 \theta$; both are +ve

$$\text{A.M} \geq \text{G.M: } \frac{9 \tan^2 \theta + 4 \cot^2 \theta}{2} \geq \sqrt{9 \tan^2 \theta \cdot 4 \cot^2 \theta} = \sqrt{36} = 6$$

$$\Rightarrow 9 \tan^2 \theta + 4 \cot^2 \theta \geq 12; \text{equality when } \tan^2 \theta = \frac{2}{3}$$

Least value = 12

Ans: 12

10. If $\cos A = \frac{15}{17}$ and $\sin B = \frac{20}{29}$, where A and B are acute, find $\cos(A + B)$. [3]

$$\cos A = \frac{15}{17} \Rightarrow \sin A = \frac{8}{17} \text{ (A acute)}$$

$$\sin B = \frac{20}{29} \Rightarrow \cos B = \frac{21}{29} \text{ (B acute)}$$

$$\cos(A + B) = \cos A \cos B - \sin A \sin B$$

$$= \frac{15}{17} \cdot \frac{21}{29} - \frac{8}{17} \cdot \frac{20}{29} = \frac{315}{493} - \frac{160}{493} = \frac{155}{493}$$

Ans: $\frac{155}{493}$

11. If $0 < A < B < \frac{\pi}{4}$, $\cos(A + B) = \frac{4}{5}$ and $\sin(A - B) = -\frac{5}{13}$, find $\tan 2B$. [5]

$$0 < A < B < \frac{\pi}{4} \Rightarrow 0 < A + B < \frac{\pi}{2} \text{ and } A - B < 0$$

$$\cos(A + B) = \frac{4}{5} \Rightarrow \sin(A + B) = \frac{3}{5}, \tan(A + B) = \frac{3}{4}$$

$$\sin(A - B) = -\frac{5}{13} \Rightarrow \cos(A - B) = \frac{12}{13}, \tan(A - B) = -\frac{5}{12}$$

$$\tan 2B = \tan[(A + B) - (A - B)] = \frac{\tan(A+B) - \tan(A-B)}{1 + \tan(A+B) \tan(A-B)}$$

$$= \frac{\frac{3}{4} - (-\frac{5}{12})}{1 - \frac{3}{4} \cdot \frac{5}{12}} = \frac{\frac{7}{6}}{\frac{11}{16}}$$

$$= \frac{56}{33}$$

Ans: $\frac{56}{33}$

12. Find the value of $\sin^2 75^\circ - \sin^2 15^\circ$. [1]

$$\sin^2 A - \sin^2 B = \sin(A + B) \sin(A - B)$$

$$= \sin 90^\circ \sin 60^\circ$$

$$= 1 \cdot \frac{\sqrt{3}}{2}$$

$$= \frac{\sqrt{3}}{2}$$

Ans: $\frac{\sqrt{3}}{2}$

13. If $A + B + C = 90^\circ$, $\tan A = \frac{1}{2}$ and $\tan B = \frac{1}{3}$, find $\tan C$. [2]

$$\text{If } A + B + C = 90^\circ: \tan A \tan B + \tan B \tan C + \tan C \tan A = 1$$

$$\frac{1}{6} + \left(\frac{1}{2} + \frac{1}{3}\right) \tan C = 1$$

$$\frac{5}{6} \tan C = \frac{5}{6} \Rightarrow \tan C = 1$$

Ans: 1

14. If $\sin A = \frac{5}{13}$ and A is acute, find $\sin 2A$ and $\tan 2A$. [2]

$$\sin A = \frac{5}{13} \Rightarrow \cos A = \frac{12}{13}, \tan A = \frac{5}{12} \text{ (A acute)}$$

$$\sin 2A = 2 \sin A \cos A = 2 \cdot \frac{5}{13} \cdot \frac{12}{13} = \frac{120}{169}$$

$$\tan 2A = \frac{2 \tan A}{1 - \tan^2 A} = \frac{\frac{10}{13}}{\frac{144}{169}} = \frac{120}{119}$$

$$\text{Ans: } \sin 2A = \frac{120}{169}, \tan 2A = \frac{120}{119}$$

15. If $\cos \alpha = -\frac{3}{5}$ and $\pi < \alpha < \frac{3\pi}{2}$, find $\cos 3\alpha$ and $\sin 2\alpha$. [5]

$$\text{Given } \cos \alpha = -\frac{3}{5} \text{ and } \alpha \in \text{III}_Q$$

$$\text{Right triangle with sides 4, 3, 5; with the signs of III}_Q: \sin \alpha = -\frac{4}{5}, \cos \alpha = -\frac{3}{5}, \tan \alpha = \frac{4}{3}$$

$$\cos 3\alpha = 4 \cos^3 \alpha - 3 \cos \alpha = 4 \left(-\frac{3}{5}\right)^3 - 3 \left(-\frac{3}{5}\right) = -\frac{108}{125} + \frac{9}{5} = \frac{117}{125}$$

$$\sin 2\alpha = 2 \sin \alpha \cos \alpha = 2 \left(-\frac{4}{5}\right) \left(-\frac{3}{5}\right) = \frac{24}{25}$$

$$\text{Ans: } \cos 3\alpha = \frac{117}{125}, \sin 2\alpha = \frac{24}{25}$$

16. Find the value of $\cos \frac{\pi}{9} + \cos \frac{3\pi}{9} + \cos \frac{5\pi}{9} + \cos \frac{7\pi}{9}$. [3]

$$\text{Angles in A.P.: } a_1 = \frac{\pi}{9}, a_n = \frac{7\pi}{9}, d = \frac{2\pi}{9}, n = 4$$

$$\text{Sum} = \frac{\cos \frac{a_1 + a_n}{2} \sin \frac{nd}{2}}{\sin \frac{d}{2}}$$

$$= \frac{\cos \frac{4\pi}{9} \sin \frac{4\pi}{9}}{\sin \frac{\pi}{9}} = \frac{\sin \frac{8\pi}{9}}{2 \sin \frac{\pi}{9}}$$

$$\sin \frac{8\pi}{9} = \sin \left(\pi - \frac{\pi}{9}\right) = \sin \frac{\pi}{9}, \text{ so the sum} = \frac{1}{2}$$

$$\text{Ans: } \frac{1}{2}$$

17. Prove that $\tan \theta \tan(60^\circ - \theta) \tan(60^\circ + \theta) = \tan 3\theta$. Hence find the value of $\tan 20^\circ \tan 40^\circ \tan 80^\circ$. [5]

$$\text{Let } t = \tan \theta: \tan(60^\circ - \theta) \tan(60^\circ + \theta) = \frac{\sqrt{3}-t}{1+\sqrt{3}t} \cdot \frac{\sqrt{3}+t}{1-\sqrt{3}t} = \frac{3-t^2}{1-3t^2}$$

$$\text{LHS} = t \cdot \frac{3-t^2}{1-3t^2} = \frac{3t-t^3}{1-3t^2} = \tan 3\theta = \text{RHS}$$

$$\text{Put } \theta = 20^\circ: \tan 20^\circ \tan 40^\circ \tan 80^\circ = \tan 60^\circ = \sqrt{3}$$

$$\text{Ans: } \sqrt{3}$$

18. If $\frac{x}{y} = \frac{\sin 105^\circ}{\sin 15^\circ}$, find $\frac{x+y}{x-y}$. [1]

$$\frac{x}{y} = \frac{\sin A}{\sin B} \Rightarrow \frac{x+y}{x-y} = \frac{\sin A + \sin B}{\sin A - \sin B} = \frac{\tan \frac{A+B}{2}}{\tan \frac{A-B}{2}}$$

$$= \frac{\tan 60^\circ}{\tan 45^\circ} = \sqrt{3}$$

$$\text{Ans: } \sqrt{3}$$

19. In $\triangle ABC$, prove that $\cos 2A + \cos 2B + \cos 2C = -1 - 4 \cos A \cos B \cos C$. [5]

$$\text{Given } A + B + C = 180^\circ, \text{ so } \sin(A + B) = \sin C \text{ and } \cos(A + B) = -\cos C$$

$$\text{LHS} = 2 \cos(A + B) \cos(A - B) + 2 \cos^2 C - 1 = -2 \cos C \cos(A - B) + 2 \cos^2 C - 1$$

$$= -1 - 2 \cos C [\cos(A - B) - \cos C] = -1 - 2 \cos C [\cos(A - B) + \cos(A + B)]$$

$$= -1 - 2 \cos C \cdot 2 \cos A \cos B = -1 - 4 \cos A \cos B \cos C$$

$$= \text{RHS}$$

$$\text{Ans: Proved: LHS} = -1 - 4 \cos A \cos B \cos C$$

20. Find the period of $3 \tan(4x - 1) + 2$. [2]

Period of $\tan(ax + b) + k$ is $\frac{\pi}{|a|}$

$$= \frac{\pi}{4}$$

Ans: $\frac{\pi}{4}$
